Chapter 5. Principles of Convection

5.1 Introduction

So far we considered heat transfer by conduction in solids in which no motion of the medium was involved. In conduction problems, the convection entered the analysis merely as a boundary condition in the form of a heat transfer coefficient. Our objective in this and the following chapters on convection is to establish the physical and mathematical basis for the understanding of convective transport and to reveal various heat transfer correlations.

The analysis of convection is complicated because the fluid motion affects the pressure drop, the drag force, and the heat transfer. To determine the drag force, or the pressure drop, the velocity field in the immediate vicinity of the surface must be known. To determine the heat transfer, the velocity distribution in the flow also is needed because the velocity enters the energy equation: the solution of the energy equation yields the temperature distribution in the flow field

In this chapter, I present a coherent view of the subject of convection in order to provide a firm basis for application. Basic concepts associated with flow over a body, flow inside a duct, and turbulence are discussed. The role of temperature and velocity distributions in the flow on heat transfer and drag force is illustrated.

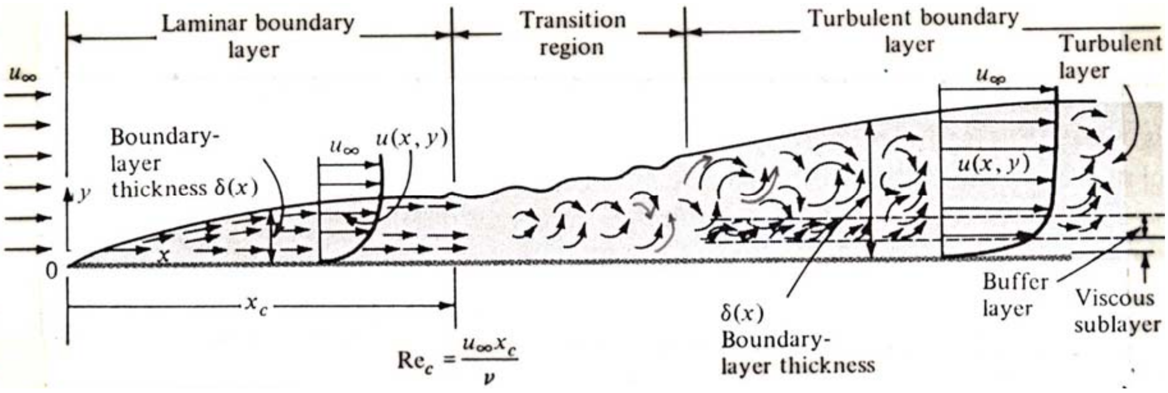
The velocity and temperature distributions are determined from the solution of the momentum and energy equation, respectively. Therefore, such equations are presented for the case of two-dimensional, constant-property, incompressible flow in rectangular and cylindrical coordinate systems. Finally, the physical significance of dimensionless parameters is discussed, and the boundary layer equations are presented.

5-2 Viscous Flow

- Flow over a Body

When a fluid flows over a body, the velocity and temperature distributions at the immediate vicinity of the surface strongly influence the heat transfer by convection. The boundary layer concept is frequently introduced to model the velocity and temperature fields near the solid surface in order to simplify the analysis of convective heat transfer. So we are concerned with two different kinds of boundary layers, the velocity boundary layer and the thermal boundary layer.

- **Velocity Boundary Layer over a Flat Plate**



The fluid at the leading edge of the plate (at x=0) has a velocity U_∞ which is parallel to the plate surface. As the fluid moves in the x-direction along the plate, those fluid particles that make contact with the surface become zero velocity (i.e., no slip condition at the wall). The region of flow which develops from the leading edge of the plate in which the effects of viscosity are observed is called the boundary layer. Therefore, starting from the plate surface there will be a retardation in the x-direction component of the velocity u. That is, $u = 0$ at $y = 0$. The retardation effect is reduced when the fluid is moving away from the plate surfaces. At distances sufficiently far from the plate the retardation effect is considered zero.

$$u \to U_\infty \text{ as } y \to \infty$$

Some arbitrary point is used to designate the $y = \delta(x)$ position where the boundary layer ends. This point is usually chosen as a distance from the surface of the plate where u equals to 99% of U_∞, that is,

$$u = 0.99\, U_\infty$$

Initially, the boundary layer development is laminar (the laminar flow remains orderly and fluid particles move along streamlines), but at some critical distance from the leading edge, depending on the flow field and fluid properties, small disturbances in the flow begin to become amplified, and a transient process takes place until the flow becomes turbulent. The turbulent flow region may be pictured as a random churning action with chunks of fluid moving to and fro in all directions.

The Reynolds number is defined for the flow over a flat plate as

$$Re_x = \frac{U_\infty x}{\nu}$$

U_∞ : freestream velocity (m/s), ν: kinematic viscosity of the fluid (m^2/s)
x: distance from the leading edge of the plate (m)

The Reynolds number is a measure of the relative magnitude of inertial force(convection) to viscous diffusion **or** the ratio of the time for viscous diffusion to occur to the time for convection to occur.

$$(Re_x)_c = \frac{U_\infty x_c}{\nu} \cong 5 \times 10^5 \text{ (a critical Reynolds number)}$$

$(Re_x)_c$ value is strongly dependent on the surface roughness, the turbulent level of the freestream, and the heat transfer(wall temperature).

$$10^5 \leqq (Re_x)_c \leqq 4\times10^6$$

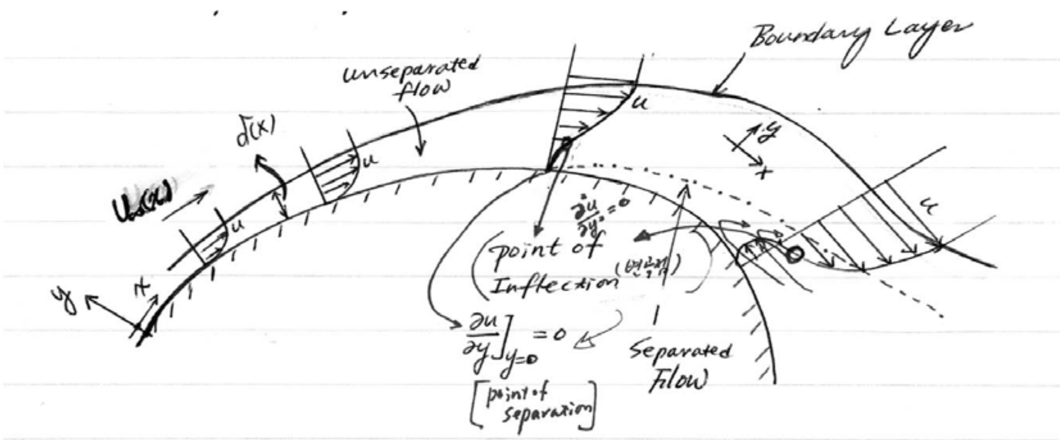
$(Re_x)_c = 10^5$ for flows with very large disturbances in the freestream and
$(Re_x)_c = 4\text{x}10^6$ for flows which are free from disturbances

In the turbulent boundary layer next to the wall, there is a very thin layer, called the viscous (or laminar) sublayer, where the flow retains its viscous (or laminar) flow character. Adjacent to the viscous sublayer is a region called the buffer layer (or overlap layer) in which there is fine-grained turbulence, and the mean axial velocity rapidly increases with the distance from the wall. The buffer layer is followed by the turbulent layer in which there is large scale turbulence, and the velocity changes relatively little with the distance from the wall.

For the flow over a flat plate, the flow field can be separated into two distinct regions (1) boundary layer region : the axial velocity component

$u(x,y)$ varies rapidly with the distance y from the plate. Hence the velocity gradients and the shear stress are considered large. (2) potential flow region, the region outside the boundary layer where the velocity gradients and shear stresses are negligible.

- **Velocity Boundary Layer along a Curved Body and Flow Separation**



The above figure shows the boundary layer concept for flow over a curved body. In this case, the x coordinate is measured along the curved surface of the body. By starting from the stagnation point and at each location, the y coordinate is measured normal to the surface of the body. The freestream velocity $U_\infty(x)$ varies with distance along the curved surface. The boundary layer concept for flow over a flat plate also applies to this particular situation. The boundary layer thickness $\delta(x)$ increases with the distance x along the surface. However, because of the curvature of the surface, after some distance x, the velocity profile $u(x,y)$ exhibits a point of separation $\left(\left. \frac{du}{dy} \right]_{y=0} = 0 \right)$ at the wall surface.

Beyond the point of inflection $\left(\frac{\partial^2 u}{\partial y^2} = 0 \right)$, the flow reversal takes place and the boundary layer is said to be detached from the wall surface (other known as "flow separation"). Beyond the point of flow reversal, the flow patterns are very complicated and the boundary layer analysis is no longer applicable.

- **Velocity Boundary Layer in a Circular duct**

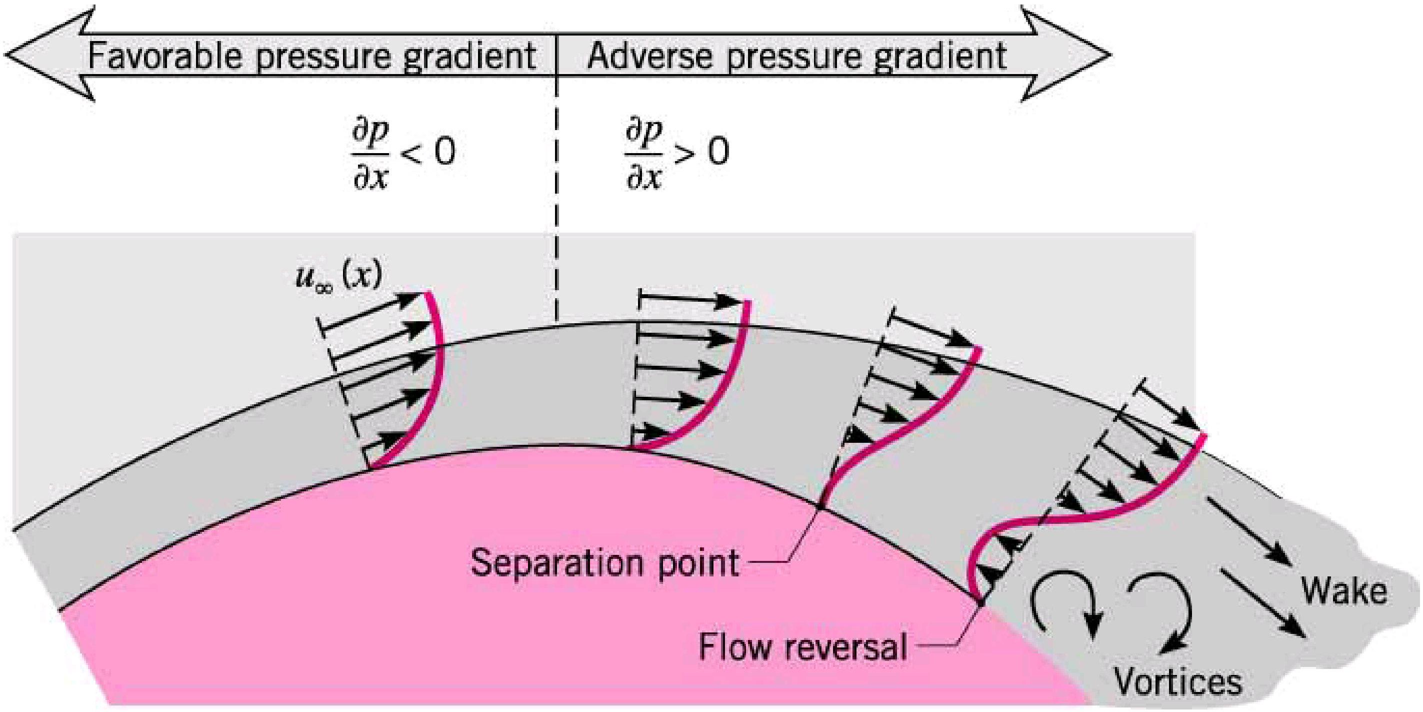


그림. 곡면 상에서의 속도경계층 형성 및 박리 현상

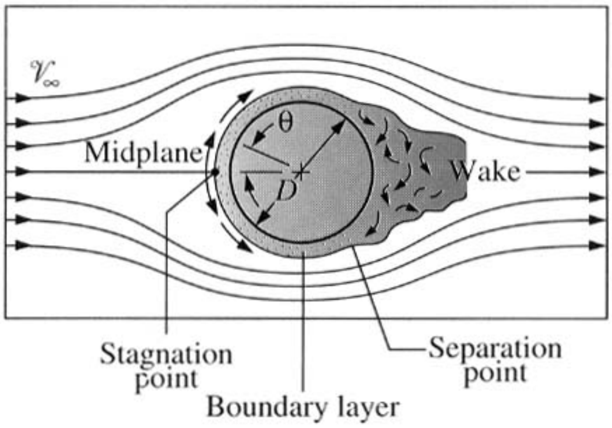
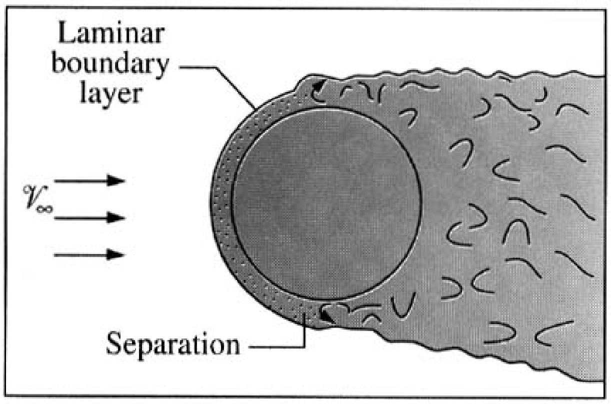
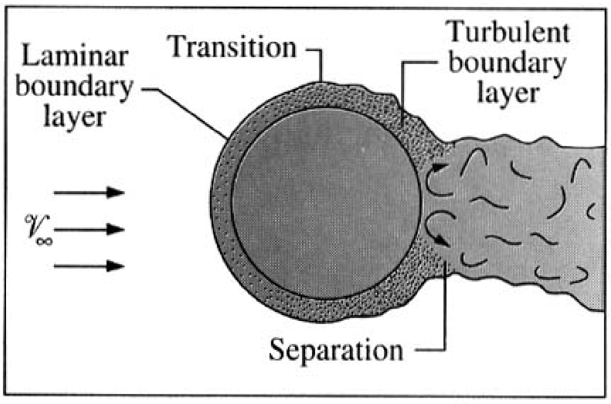


그림 11. 원통에 교차하는 대표적인 유동의 형태



(*a*) Laminar flow (Re $< 2 \times 10^5$)



(*b*) Turbulence occurs (Re $> 2 \times 10^5$)

그림 12. 난류가 유동의 박리를 지연시킨다

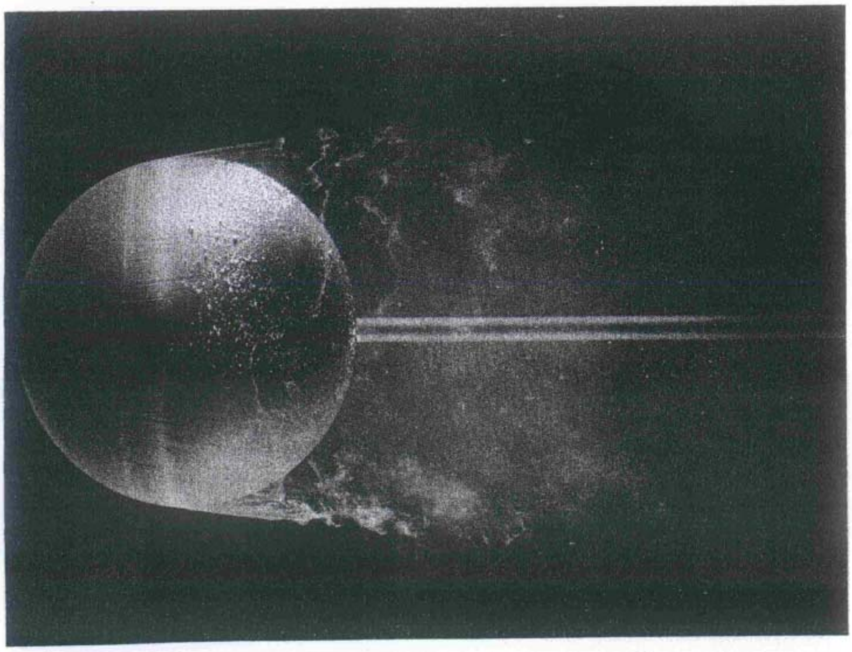


Figure 15-20 Subcritical flow over a sphere is shown at Re = 15,000. Laminar separation occurs forward of the equator. Photograph courtesy of H. Werlé, ONERA, Catillon, France.

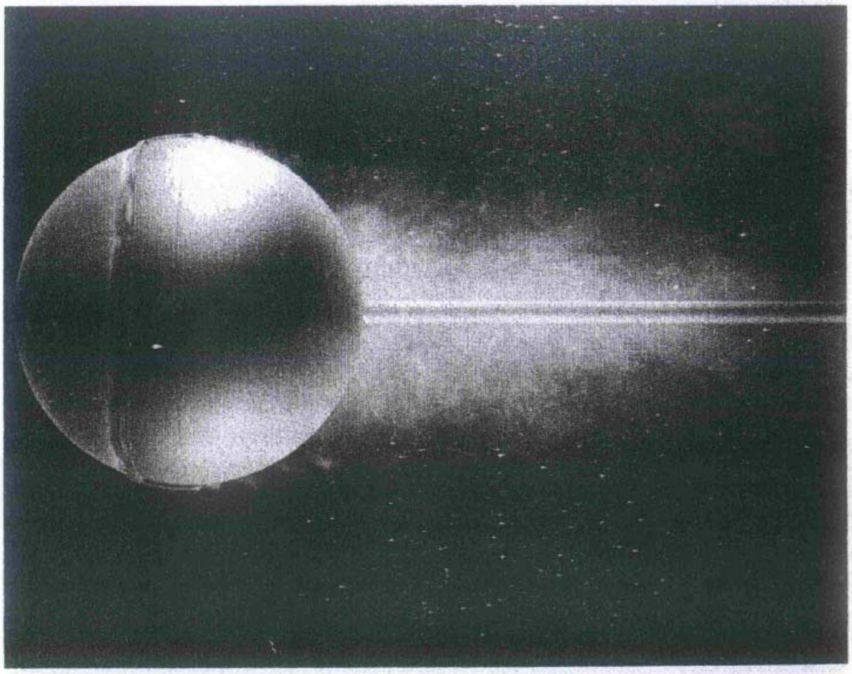


Figure 15-21 Supercritical flow at Re = 30,000. Normally this flow is subcritical, but a small trip wire has induced transition to a turbulent boundary layer. Separation is now downstream of the equator, and the wake is smaller. Photograph from ONERA by H. Werlé (1980).

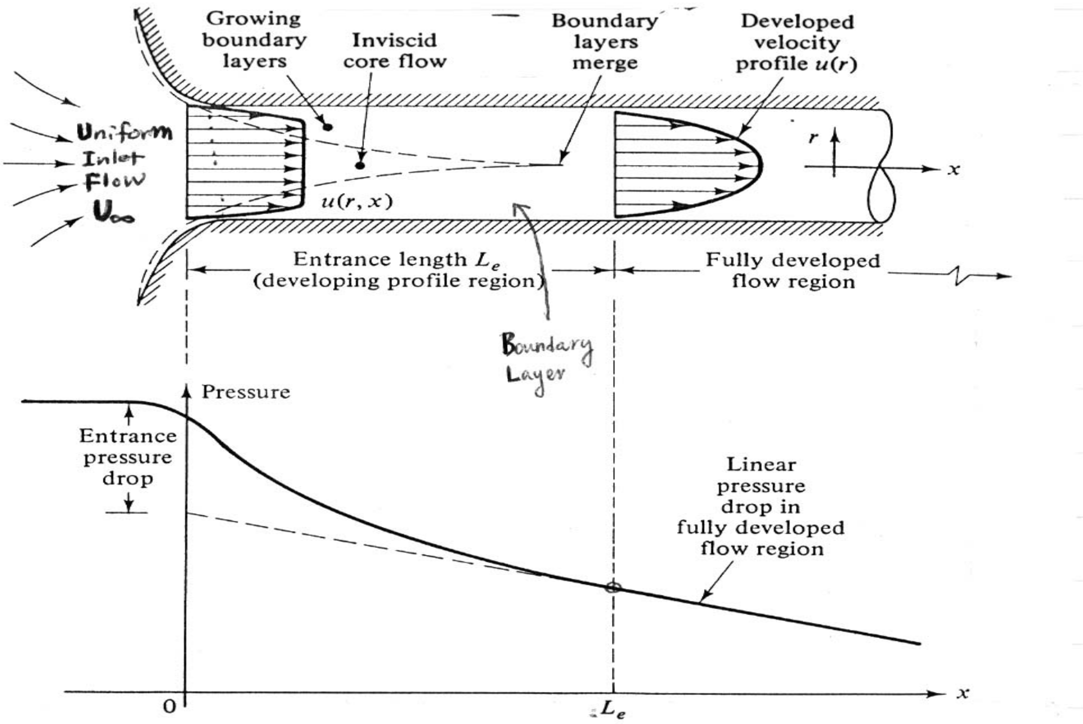
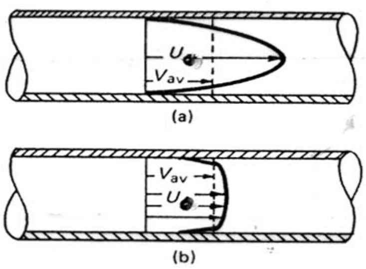


Fig. 1 Developing velocity profiles and pressure changes in the entrance of a duct flow.



U_o: the centerline (maximum) velocity

V_{av}: the average velocity

Fig. 2 Comparison of laminar and turbulent pipe-flow velocity profiles for the same volumetric flow rate: (a) laminar flow; (b) turbulent flow.

Table. U_{av}/U_o for turbulent flow in a circular duct

$\frac{U_{av}}{U_o} \cong (1+1.33\sqrt{f})^{-1}$ $\quad f = 0.316\, Re_p^{-\frac{1}{4}}$ for $[4000 \leqq Re_D \leqq 10^5]$						
Re_D	4000	10^4	10^5	10^6	10^7	10^8
$\frac{U_{av}}{U_o}$	0.79	0.811	0.849	0.875	0.893	0.907

Figures in the previous page show a viscous flow in a circular duct. The flow is constrained by the boundary walls, and the viscous effects will grow and meet and permeate the entire flow. There is an entrance region where a nearly inviscid upstream flow converges and enters the tube. Viscous boundary layers grow downstream, retarding the axial flow $u(r,x)$ at the wall and thereby accelerating the center-core flow to maintain the incompressible continuity requirement. $\dot{V} = \int u\,dA = const$

At a finite distance from the entrance the boundary layers merge and the inviscid core disappears. The tube flow is then entirely viscous, and the axial velocity adjusts slightly further until at $x = L_e$. It no longer changes with x and is said to be "fully developed", $u \approx u(r)$ only. Downstream of $x = L_e$ the velocity profile is constant, the wall shear stress is constant, and the pressure drops linearly with x.

As illustrated in Figure, the velocity profile for turbulent flow in the pipe is markedly different from that for laminar flow. The two profiles are for the same volumetric flow rate. We see that the parabolic velocity profile of laminar flow has a larger value for the maximum velocity(U_o), the velocity at the centerline, but a lower velocity gradient at wall, that is, less shear stress at the wall according to the shear stress equation,

$$\tau_{wall} = \mu \frac{du}{dy}\bigg]_{wall(y=0)}$$

Conversely, for turbulent flow, the velocity increases rapidly with distance from the wall from the no-slip condition at the wall (i.e., $U_{r=R} = 0$). Thus, a turbulent pipe flow produces relatively large shear stress. The turbulent shear stress can be **<u>hundreds of times</u>** greater than the laminar shear stress due to the mixing motion (a sliding of the one particle layer over another) of the fluid. Furthermore, with the large number of random particle fluctuations present in a turbulent flow, there is a tendency toward mixing of the fluid and a more uniform velocity profile. The interchange of momentum between faster- and slower-moving particles tends to "even out" the velocity profile. Thus, for turbulent pipe flow, the time-averaged mean velocity plotted versus radius might appear much more uniform, except very near the wall(laminar sublayer), where the large velocity gradient produces relatively large shear forces and a correspondingly large head loss (pressure loss).

Another way of looking at the difference between laminar and turbulent flows is to consider what happens when a small disturbance is introduced into a flow. If the flow is laminar, a small disturbance is damped out by viscous forces. If the disturbance cannot be damped out, but continues to grow and affect the entire stream, we have turbulent flow.

Even in turbulent pipe flow, with the great majority of the flow characterized by rough, irregular motions, there will always be a thin layer of smooth laminar flow near a wall, for the particle fluctuations must die out near a boundary. This thin layer is called laminar sublayer (or viscous sublayer). The thickness of this layer depends on the degree of turbulence of the main stream - the more turbulent the flow, the thinner the sublayer. In any case, the thickness of this layer is only a very small fraction of the pipe diameter.

In a pipe flow, the Reynolds number is again used as a criterion for laminar and turbulent flow. The generally accepted value for the transition from laminar to turbulent flow is

$$(Re_D)_c = \frac{U_{av} D}{\nu} \cong 2{,}300$$

The transition Reynolds number may change depending on the pipe roughness and smoothness of the flow as follows:

$$2{,}000 < (Re_D)_c < 4{,}000$$

However, the actual value must depend to a certain extent on the magnitude of the disturbances introduced into the flow by such factors as the pipe inlet, pipe bends, and extraneous vibrations due to the proximity of pumping machinery.

(Note) $Re_D = \dfrac{U_{av} D}{\nu} = \dfrac{\rho U_{av} D}{\mu} = \dfrac{4\dot{m}}{\pi \mu D} = \dfrac{4\dot{\mathrm{V}}}{\pi \nu D}$

where, $\dot{m}$: mass flow rate (kg/s), $\dot{\mathrm{V}}$: volume flow rate (m^3/s)

Mass velocity (kg/m^2 · s), $G = \frac{\dot{m}}{A} = \rho U_{av}$

The entrance length, L_e, required to attain fully developed flow is dependent on the type of flow. For laminar flow, the required entrance length is given approximately by: $\frac{L_e}{D} \cong 0.057\, Re_D$ – **Laminar Flow**

Thus, the maximum laminar entrance length, at $(Re_D)_{tr} = 2{,}300$, is $L_e = 131\, D$, which is the longest development length possible. In turbulent flow the boundary layers grow much faster, and L_e is relatively shorter according to the approximation

$$\frac{L_e}{D} \cong 4.4\,(Re_D)^{1/6} \quad \text{or} \quad 1.359\,(Re_D)^{1/4}$$ – **Turbulent Flow**

The usual entrance length for turbulent flow is between 25 and 60 pipe diameters, the value depending on the wall roughness and inlet shape. For instance, a square-edged opening requires shorter entrance length than a rounded-edged opening does.

Some Computed Entrance Lengths for Turbulent Flow						
Re_D	4000	10^4	10^5	10^6	10^7	10^8
L_e/D	18	20	30	44	65	95

5-3 | INVISCID FLOW

Although no real fluid is inviscid, in some instances the fluid may be treated as such, and it is worthwhile to present some of the equations that apply in these circumstances. For example, in the flat-plate problem discussed above, the flow at a sufficiently large distance from the plate will behave as a nonviscous flow system. The reason for this behavior is that the velocity gradients normal to the flow direction are very small, and hence the viscous-shear forces are small.

If a balance of forces is made on an element of incompressible fluid and these forces are set equal to the change in momentum of the fluid element, the Bernoulli equation for flow along a streamline results:

$$\frac{p}{\rho} + \frac{1}{2}\frac{V^2}{g_c} = \text{const} \qquad \textbf{[5-7}\boldsymbol{a}\textbf{]}$$

or, in differential form,

$$\frac{dp}{\rho} + \frac{V\,dV}{g_c} = 0 \qquad \textbf{[5-7}\boldsymbol{b}\textbf{]}$$

where

ρ = fluid density, kg/m^3

p = pressure at particular point in flow, Pa

V = velocity of flow at that point, m/s

The Bernoulli equation is sometimes considered an energy equation because the $V^2/2g_c$ term represents kinetic energy and the pressure represents potential energy; however, it must be remembered that these terms are derived on the basis of a dynamic analysis, so that the equation is fundamentally a dynamic equation. In fact, the concept of kinetic energy is based on a dynamic analysis.

When the fluid is compressible, an energy equation must be written that will take into account changes in internal thermal energy of the system and the corresponding changes in temperature. For a one-dimensional flow system this equation is the steady-flow energy equation for a control volume,

$$i_1 + \frac{1}{2g_c}V_1^2 + Q = i_2 + \frac{1}{2g_c}V_2^2 + Wk \qquad \textbf{[5-8]}$$

where i is the enthalpy defined by

$$i = e + pv \qquad \textbf{[5-9]}$$

and where

e = internal energy

Q = heat added to control volume

Wk = net external work done in the process

v = specific volume of fluid

(The symbol i is used to denote the enthalpy instead of the customary h to avoid confusion with the heat-transfer coefficient.) The subscripts 1 and 2 refer to entrance and exit conditions to the control volume. To calculate pressure drop in compressible flow, it is necessary to specify the equation of state of the fluid, for example, for an ideal gas,

$$p = \rho RT \quad \Delta e = c_v \Delta T \quad \Delta i = c_p \Delta T$$

The gas constant for a particular gas is given in terms of the universal gas constant $\Re$ as

$$R = \frac{\Re}{M}$$

where M is the molecular weight and $\Re = 8314.5$ J/kg · mol · K. For air, the appropriate ideal-gas properties are

$$R_{\text{air}} = 287 \text{ J/kg} \cdot \text{K} \quad c_{p,\text{air}} = 1.005 \text{ kJ/kg} \cdot {}^\circ\text{C} \quad c_{v,\text{air}} = 0.718 \text{ kJ/kg} \cdot {}^\circ\text{C}$$

To solve a particular problem, we must also specify the process. For example, reversible adiabatic flow through a nozzle yields the following familiar expressions relating the properties at some point in the flow to the Mach number and the stagnation properties, i.e., the properties where the velocity is zero:

$$\frac{T_0}{T} = 1 + \frac{\gamma - 1}{2} M^2$$

$$\frac{p_0}{p} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\gamma/(\gamma-1)}$$

$$\frac{\rho_0}{\rho} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{1/(\gamma-1)}$$

where

$T_0, p_0, \rho_0 =$ stagnation properties
$\gamma =$ ratio of specific heats c_p/c_v
$M =$ Mach number

$$M = \frac{V}{a}$$

where a is the local velocity of sound, which may be calculated from

$$a = \sqrt{\gamma g_c RT} \qquad \textbf{[5-10]}$$

for an ideal gas.† For air behaving as an ideal gas this equation reduces to

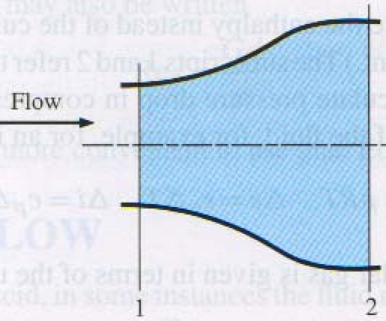
$$a = 20.045\sqrt{T} \quad \text{m/s} \qquad \textbf{[5-11]}$$

where T is in degrees Kelvin.

EXAMPLE 5-1 Water Flow in a Diffuser

Water at 20°C flows at 8 kg/s through the diffuser arrangement shown in Figure Example 5-1. The diameter at section 1 is 3.0 cm, and the diameter at section 2 is 7.0 cm. Determine the increase in static pressure between sections 1 and 2. Assume frictionless flow.

Figure Example 5-1



■ **Solution**

The flow cross-sectional areas are

$$A_1 = \frac{\pi d_1^2}{4} = \frac{\pi(0.03)^2}{4} = 7.069 \times 10^{-4}\ \text{m}^2$$

$$A_2 = \frac{\pi d_2^2}{4} = \frac{\pi(0.07)^2}{4} = 3.848 \times 10^{-3}\ \text{m}^2$$

†The isentropic flow formulas are derived in Reference 7, p. 629.

The density of water at 20°C is 1000 kg/m^3, and so we may calculate the velocities from the mass-continuity relation

$$u = \frac{\dot{m}}{\rho A}$$

$$u_1 = \frac{8.0}{(1000)(7.069 \times 10^{-4})} = 11.32\ \text{m/s} \quad [37.1\ \text{ft/s}]$$

$$u_2 = \frac{8.0}{(1000)(3.848 \times 10^{-3})} = 2.079\ \text{m/s} \quad [6.82\ \text{ft/s}]$$

The pressure difference is obtained from the Bernoulli equation (5-7*a*):

$$\frac{p_2 - p_1}{\rho} = \frac{1}{2g_c}(u_1^2 - u_2^2)$$

$$p_2 - p_1 = \frac{1000}{2}[(11.32)^2 - (2.079)^2]$$

$$= 61.91\ \text{kPa} \quad [8.98\ \text{lb/in}^2\ \text{abs}]$$

Isentropic Expansion of Air

EXAMPLE 5-2

Air at 300°C and 0.7 MPa pressure is expanded isentropically from a tank until the velocity is 300 m/s. Determine the static temperature, pressure, and Mach number of the air at the high-velocity condition. $\gamma = 1.4$ for air.

■ Solution

We may write the steady-flow energy equation as

$$i_1 = i_2 + \frac{u_2^2}{2g_c}$$

because the initial velocity is small and the process is adiabatic. In terms of temperature,

$$c_p(T_1 - T_2) = \frac{u_2^2}{2g_c}$$

$$(1005)(300 - T_2) = \frac{(300)^2}{(2)(1.0)}$$

$$T_2 = 255.2^\circ\mathrm{C} = 528.2\ \mathrm{K} \quad [491.4^\circ\mathrm{F}]$$

We may calculate the pressure from the isentropic relation

$$\frac{p_2}{p_1} = \left(\frac{T_2}{T_1}\right)^{\gamma/(\gamma-1)}$$

$$p_2 = (0.7)\left(\frac{528.2}{573}\right)^{3.5} = 0.526\ \mathrm{MPa} \quad [76.3\ \mathrm{lb/in^2\ abs}]$$

The velocity of sound at condition 2 is

$$a_2 = (20.045)(528.2)^{1/2} = 460.7\ \mathrm{m/s} \quad [1511\ \mathrm{ft/s}]$$

so that the Mach number is

$$M_2 = \frac{u_2}{a_2} = \frac{300}{460.7} = 0.651$$

5-4 Laminar Boundary Layer on a Flat Plate

- Conservation of mass (or continuity Equation)

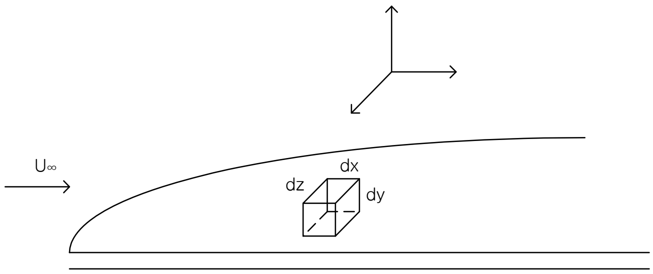
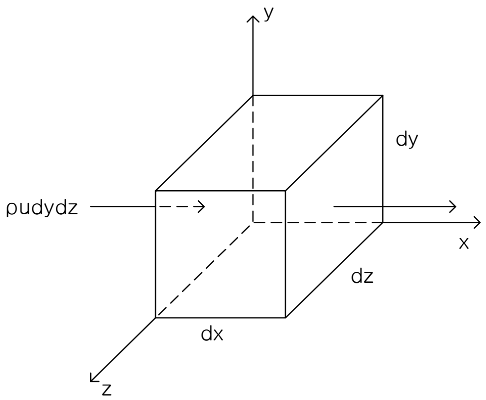


Figure. Elemental control volume for momentum balance and mass balance on laminar boundary layer.

Examine an elemental control volume ($dx\,dy\,dz$) in the shape of a rectangular parallel pipe fixed in x y z. Showing the inlet and outlet mass flows on the x faces.



$$[\rho u + \frac{\partial}{\partial x}(\rho u)\,dx]\,dy\,dz$$

(Note) If $\rho u]_1$ (mass velocity on the left face) is known, $\rho u]_2$ (mass velocity on the right face) can be expressed as a Taylor series about point x=0 and retaining only first-order differentials.

$$\begin{aligned}\rho u]_2 &= \rho u]_1 + \frac{\partial}{\partial x}(\rho u)_1 \frac{dx}{1!} + \frac{\partial^2(\rho u)_1}{\partial x^2}\frac{dx^2}{2!} + \cdots \\ &= \rho u + \frac{\partial(\rho u)}{\partial x}dx\end{aligned}$$

∴ The net rate of mass flow through x faces is : $\frac{\partial}{\partial x}(\rho u)\,dx\,dy\,dz$

Likely, for y and z faces

$\frac{\partial}{\partial y}(\rho v)\,dx\,dy\,dz$, $\frac{\partial}{\partial z}(\rho w)\,dx\,dy\,dz$, respectively.

(Note) : u, v, w are x, y, and z component of velocity, respectively.

Thus, the net rate of mass outflux through the control surface (boundary of the control volume) is :

$$\left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right]dx\,dy\,dz \qquad \text{———} \quad (1)$$

and the rate of decrease of mass inside control volume is,

$$-\frac{\partial}{\partial t}(\rho\,dx\,dy\,dz) = -\frac{\partial \rho}{\partial t}\,dx\,dy\,dz \qquad \text{———} \quad (2)$$

Equate (1) with (2) and cancel out $dx\,dy\,dz$

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = -\frac{\partial \rho}{\partial t} \qquad \text{———} \quad (3)$$

(differential continuity equation)

To simplify the analysis for the boundary layer we assume the followings

① The fluid is incompressible, constant-property and Newtonian fluid

② The flow is steady and two-dimensional

Eq. (3) reduces to $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ ——— (4)

where, $u = u(x,y)$, $v = v(x,y)$

Continuity equation in rectangular coordinates for the steady, two-dimensional flow of an incompressible constant-property, Newtonian fluid.

In the cylindrical coordinate system, the continuity equation for the steady, two-dimensional flow of an incompressible constant-property, Newtonian fluid is

$$\frac{1}{r}\frac{\partial(rv)}{\partial r}+\frac{\partial u}{\partial z}=0$$

where, $u=u(r,z)\rightarrow z\ direction\,velocity,\quad v=v(r,z)\rightarrow r\ direction\ \ velocity$

- **Conservation of Momentum (Momentum Equation)**

We use the same elemental control volume $(dxdydz)$ as for the derivation of the continuity equation

The momentum equation is derived from Newton's second law of motion, which states that mass times the acceleration in a given direction is proportional to the external forces acting on the body in the same direction.

$$\vec{F}=m\vec{a}=\rho v\frac{d\vec{V}}{dt} \qquad \text{———————} \quad (5)$$

The external forces acting on a volume element consist of the body force (F_b) and the surface forces(F_s) ;

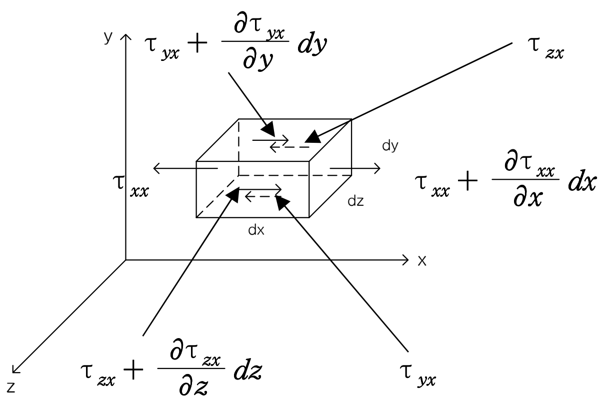
$$\vec{F}=\vec{F_b}+\vec{F_s}$$

① F_b : Acting on the material inside the control volume (the weight due to an action of gravity - $\rho dx\,dy\,dz\,\vec{g}$)

② F_s : Acting on the control surface (due to hydrostatic pressure and viscous shear stresses)

※ Hydrostatic pressure gives rise to a driving action to cause the fluid to flow

※ Viscous shear stresses stem from the motion of fluid.



where, τ_{ij} is the value of shear stress acting on a plane whose normal is parallel to the i-direction and the stress itself is parallel to the j-direction.

In order to simplify the analysis we again assume:

① There are no pressure variations in the direction perpendicular to the flat plate

② The fluid is incompressible and the flow is steady, two-dimensional

③ The viscosity is constant

④ Viscous-shear forces in the y-direction are negligible

⑤ Only x-direction momentum is evaluated because the forces considered in the analysis are those in the x-direction

Now, let's consider the external forces first.

The body force is assumed to be negligibly small : $F_{b(x)} = 0$

The net surface forces in the x-direction are as follows;

$$F_{s(x)} = (\tau_{xx} + \frac{\partial \tau_{xx}}{\partial x} dx)\, dy\, dz - \tau_{xx} dy\, dz + (\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} dy)\, dx\, dz - \tau_{yx} dx\, dz$$

$$+ (\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z} dz)\, dx\, dy - \tau_{zx} dx\, dy \qquad \text{——— (6)}$$

$$= \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \right) dx\, dy\, dz$$

The x-direction acceleration of $\rho v \frac{d\vec{V}}{dt}$ in Eq (5) becomes

$$\rho v \frac{du}{dt} = \rho\, dx\, dy\, dz \left[\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right] \qquad \text{——— (7)}$$

$$= \rho\, dx\, dy\, dz \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right)$$

Now, equate (6) and (7)

$$\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} = \rho \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right)$$

where, $\tau_{xx} = -p + 2\mu \frac{\partial u}{\partial x}$, $\tau_{yx} = \tau_{xy} = \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)$

$$\tau_{zx} = \tau_{xz} = \mu \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) = 0$$

Then, $\frac{\partial}{\partial x}\left(-p + 2\mu \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) = \rho \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right)$

$$\Rightarrow \rho \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = -\frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \qquad \text{——— (8)}$$

$\Rightarrow$ **the x-momentum equation for the steady, 2-dimensional laminar flow of an incompressible fluid**

For the boundary layer flow with constant properties, $p = p(x)$ only and

$\frac{\partial^2 u}{\partial x^2} << \frac{\partial^2 u}{\partial y^2}$ (because the boundary layer δ is very thin)

$$\therefore \rho \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = -\frac{dp}{dx} + \mu \frac{\partial^2 u}{\partial y^2} \qquad \text{——— (9)}$$

$\Rightarrow$ **the x-momentum equation of the laminar boundary layer**

For the boundary flow, the equivalent of the momentum equation (8) in the two-dimensional (r, z) cylindrical coordinate system is ;

$$z\text{-momentum ; } \rho\left(v\frac{\partial u}{\partial r}+u\frac{\partial u}{\partial z}\right)=-\frac{dp}{dx}+\mu\left(\frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right)+\frac{\partial^2 u}{\partial z^2}\right)$$

The momentum equation (9) contains the pressure-gradient term $\frac{dp}{dx}$. The pressure $p(x)$ is imposed upon the boundary layer from the external free-stream $U_\infty(x)$ outside the boundary layer, where x is the coordinate parallel to the wall (flat plate or curved surface body). The pressure $p(x)$ is related to $U_\infty(x)$ by Bernoulli's equation.

$\frac{P(x)}{\rho}+\frac{U(x)_\infty^2}{2}=C$ $\Rightarrow$ take a derivative of this equation with respect to x to obtain the following.

$$-\frac{dp(x)}{dx}=\rho\, U_\infty(x)\frac{dU_\infty(x)}{dx} \qquad \text{——— (10)}$$

Eq. (10) relates the pressure-gradient term to the external stream's velocity $U_\infty(x)$, which is assumed to be available from the solution of the potential flow (an inviscid outer flow). Thus, $\frac{dp}{dx}$ term in Eq. (9) is a known quantity. In the case of flow over a flat plate with uniform free-stream velocity U_∞ then $\frac{dp}{dx}=0$, because $\frac{dU_\infty(x)}{dx}$ in Eq. (10) becomes zero. Therefore, the momentum boundary equation for flow over a flat plate becomes;

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=\mu\frac{\partial^2 u}{\partial y^2} \qquad \text{——— (11)}$$

- **Laminar Boundary Layer Thickness on a Flat Plate**

By making a momentum-and-force balance on the control volume bounded by the planes 1-A, 2-A, A-A, and the solid wall, the Von Kármán integral boundary layer equation is obtained as

$$\mu \frac{\partial u}{\partial y}\bigg]_{y=0} = \tau_w = \rho \frac{d}{dx}\int_0^\delta (U_\infty - u)u\,dy \qquad \text{———} \quad (12)$$

⇒ Von Kármán momentum integral approximation

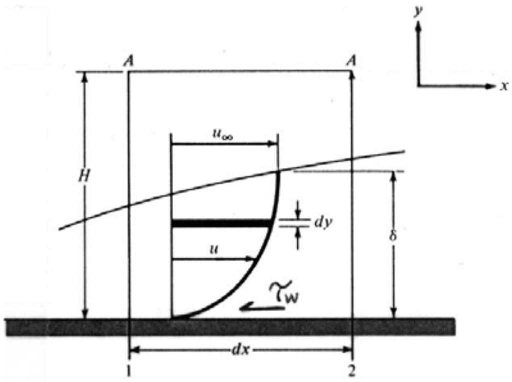


Fig. 5-5 Elemental control volume for integral momentum analysis of laminar boundary layer.

If the velocity profile, *u*, is known, the appropriate function could be inserted into Eq. (12) to obtain an expression for the boundary layer thickness. For an approximate analysis, we write down some conditions which the velocity function must satisfy ;

① $u = 0 \; at \; y = 0$

② $u = U_\infty \; at \; y = \delta$

③ $\frac{\partial u}{\partial y} = 0 \; at \; y = \delta$

④ for a constant pressure condition $\frac{\partial^2 u}{\partial y^2} = 0 \; at \; y = 0$ from Eq. (11) since $u = v = 0 \; at \; y = 0$ (no slip condition)

The simplest function that satisfies these four conditions is a third-degree polynomial approximation.

$u(x,y) = C_1 + C_2 y + C_3 y^2 + C_4 y^3$ for $0 \leq y \leq \delta(x)$

The application of four conditions results in a velocity profile in the form

$$\frac{u(x,y)}{U_\infty} = \frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3 \qquad \text{———} \quad (13)$$

Now, the velocity profile is introduced into Eq. (12)

$$\frac{d}{dx}\left\{\rho\ U_\infty{}^2\int_0^\delta\left[\frac{3}{2}\frac{y}{\delta}-\frac{1}{2}\left(\frac{y}{\delta}\right)^3\right]\left[1-\left[\frac{3}{2}\frac{y}{\delta}+\frac{1}{2}\left(\frac{y}{\delta}\right)^3\right]dy\right\} = U_\infty\mu\frac{\partial}{\partial y}\left\{\frac{3}{2}\frac{y}{\delta}-\frac{1}{2}\left(\frac{y}{\delta}\right)^3\right\}_{y=0}$$

$\Rightarrow \frac{d}{dx}\left(\frac{39}{280}\rho\ U_\infty{}^2\delta\right) = \frac{3}{2}\frac{\mu U_\infty}{\delta}$ (separate variables δ, x and integrate term)

$\frac{\delta^2}{2} = \frac{140}{13}\frac{\nu x}{U_\infty} + C$ (apply a condition at a leading edge: $\delta = 0\ at\ x = 0$)

Finally, we obtain the laminar boundary layer thickness on a flat plate.

$\frac{\delta}{x} = 4.64\,(Re_x)^{-\frac{1}{2}}$ $\Rightarrow$ **approximate solution**

On the other hand, an exact solution for the laminar boundary layer thickness on a flat plate has been given by Blasius who transformed the boundary layer equations to a third-order ordinary differential equation and solved it by a series expansion method and its result is:

$\frac{\delta}{x} = 5.0\,(Re_x)^{-\frac{1}{2}}$ $\Rightarrow$ **exact solution**

We have checked our approximate method against the exact solution for the special case of a zero pressure gradient(flat plate) with good success. We may now, with confidence, use this approximate procedure for situations where we do not have a zero pressure gradient (flow over a curved body). By using curvilinear coordinate it is possible to extend this method of procedure to boundaries of mild curvature.

Mass Flow and Boundary-Layer Thickness

EXAMPLE 5-3

Air at 27°C and 1 atm flows over a flat plate at a speed of 2 m/s. Calculate the boundary-layer thickness at distances of 20 cm and 40 cm from the leading edge of the plate. Calculate the mass flow that enters the boundary layer between $x = 20$ cm and $x = 40$ cm. The viscosity of air at 27°C is 1.85×10^{-5} kg/m · s. Assume unit depth in the z direction.

■ Solution

The density of air is calculated from

$$\rho = \frac{p}{RT} = \frac{1.0132 \times 10^5}{(287)(300)} = 1.177\ \text{kg/m}^3 \quad [0.073\ \text{lb}_m/\text{ft}^3]$$

The Reynolds number is calculated as

$$\text{At } x = 20\ \text{cm:} \qquad \text{Re} = \frac{(1.177)(2.0)(0.2)}{1.85 \times 10^{-5}} = 25{,}448$$

$$\text{At } x = 40\ \text{cm:} \qquad \text{Re} = \frac{(1.177)(2.0)(0.4)}{1.85 \times 10^{-5}} = 50{,}897$$

The boundary-layer thickness is calculated from Equation (5-21):

$$\text{At } x = 20\ \text{cm:} \qquad \delta = \frac{(4.64)(0.2)}{(25{,}448)^{1/2}} = 0.00582\ \text{m} \quad [0.24\ \text{in}]$$

$$\text{At } x = 40\ \text{cm:} \qquad \delta = \frac{(4.64)(0.4)}{(50{,}897)^{1/2}} = 0.00823\ \text{m} \quad [0.4\ \text{in}]$$

To calculate the mass flow that enters the boundary layer from the free stream between $x = 20$ cm and $x = 40$ cm, we simply take the difference between the mass flow in the boundary layer at these two x positions. At any x position the mass flow in the boundary layer is given by the integral

$$\int_0^\delta \rho u\, dy$$

where the velocity is given by Equation (5-19),

$$u = u_\infty \left[\frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right]$$

Evaluating the integral with this velocity distribution, we have

$$\int_0^\delta \rho u_\infty \left[\frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right] dy = \frac{5}{8}\rho u_\infty \delta$$

Thus the mass flow entering the boundary layer is

$$\begin{aligned}\Delta m &= \tfrac{5}{8}\rho u_\infty(\delta_{40} - \delta_{20}) \\ &= (\tfrac{5}{8})(1.177)(2.0)(0.0082 - 0.0058) \\ &= 3.531 \times 10^{-3}\ \text{kg/s} \quad [7.78 \times 10^{-3}\ \text{lb}_m/\text{s}]\end{aligned}$$

5-5 Energy equation of the boundary layer

The temperature distribution in the flow field is governed by the energy equation, which can be derived by writing an energy balance according to the first Law of thermodynamic for a differential volume element in the flow field.

If there are no distributed energy sources in the fluid, the energy balance on a differential volume element may be stated as

Rate of energy input due to conduction	+	rate of energy input due to work done by body forces	+	rate of energy input due to work done by surface stresses	=	rate of increase of energy in the element

To derive the energy equation, each term in this expression should be evaluated. Here we consider the energy equation in the rectangular coordinate system for steady, two-dimensional (x,y) flow of an incompressible, constant-property, Newtonian fluid. Let $\Delta X \Delta Y \cdot 1$ be the differential volume element about a point(x,y) in the flow field. Various terms in the above Eq. are evaluated now.

First, the heat addition into the element $\Delta X \Delta Y \cdot 1$ by conduction occurs in the x and y directions. Referring to the nomenclature shown in Fig. 6-13, we write

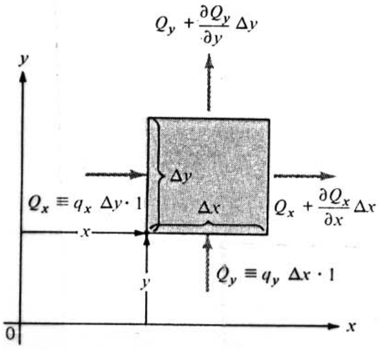
Rate of energy addition by conduction
$$= -\left(\frac{\partial Q_x}{\partial x}\Delta x + \frac{\partial Q_y}{\partial y}\Delta y\right) = -\left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y}\right)\Delta x \Delta y$$

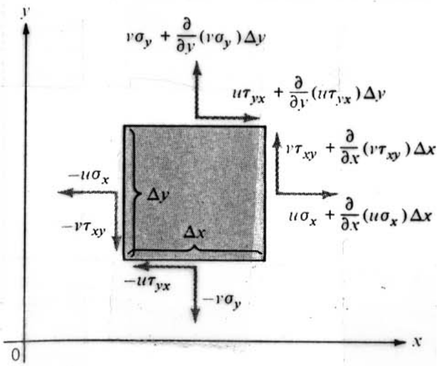
$$= -k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)\Delta x \Delta y 1$$

since $q_x = -k\frac{\partial T}{\partial x}$ and $q_y = -k\frac{\partial T}{\partial y}$

Second, if F_x and F_y are the body forces acting per unit volume of the element and u and v are the velocity components in the x and y directions, respectively, the energy input into the volume element $\Delta X \Delta Y \cdot 11$ resulting from the increase in potential energy becomes

Rate of energy input by body forces $= (uF_x + vF_y)\Delta x \Delta y 1$





Third, the rate of energy input to the volume element $\Delta X \Delta Y \cdot 1$ due to surface stress consists of the contributions from the stresses $\sigma_x, \sigma_y, \tau_{yx}$, and τ_{xy}. By referring to the illustration and nomenclature in Fig.6-14, the energy input due to the normal stress σ_x is given by

$$\left\{-u\sigma_x + \left[u\sigma_x + \frac{\partial}{\partial x}(u\sigma_x)\Delta x\right]\right\}\Delta y 1 = \Delta x \Delta y 1 \frac{\partial}{\partial x}(u\sigma_x)$$

and, due to the normal stress σ_y, given by

$$\left\{-v\sigma_y + \left[v\sigma_y + \frac{\partial}{\partial y}(v\sigma_y)\Delta y\right]\right\}\Delta x 1 = \Delta x \Delta y 1 \frac{\partial}{\partial y}(v\sigma_y)$$

Similarly, the energy input due to the stresses τ_{yx}, and τ_{xy} are given, respectively, by

$$-u\tau_{yx} + \left[u\tau_{yx} + \frac{\partial}{\partial y}(u\tau_{yx})\Delta y\right]\Delta x 1 = \Delta x \Delta y 1 \frac{\partial}{\partial y}(u\tau_{yx})$$

$$-v\tau_{xy} + \left[v\tau_{xy} + \frac{\partial}{\partial x}(v\tau_{xy})\Delta x\right]\Delta y 1 = \Delta x \Delta y 1 \frac{\partial}{\partial x}(v\tau_{xy})$$

The total rate of energy input into the element due to the stress is obtained by summing the above four quantities:

$$\begin{array}{l}\text{Rate of energy input}\\ \text{by surface stresses}\end{array} = \left[\frac{\partial}{\partial x}(u\sigma_x) + \frac{\partial}{\partial y}(v\sigma_y) + \frac{\partial}{\partial y}(u\tau_{yx}) + \frac{\partial}{\partial x}(v\tau_{xy})\right]\Delta x \Delta y 1$$

Fourth the energy contained in the volume element is considered to consist of the specific internal energy e per unit mass and the kinetic energy $\frac{1}{2}(u^2+v^2)$ per unit mass of the fluid. Then the energy content of the volume element $\Delta X \Delta Y \cdot 1$ becomes

$$\rho\left[e+\frac{1}{2}(u^2+v^2)\right]\Delta x\Delta y 1$$

The rate of increase of this energy is obtained by taking its total derivative, that is,

$$\begin{array}{l}\text{Rate of increase of}\\ \text{energy of element}\end{array} = \rho\left[\frac{De}{Dt}+\frac{1}{2}\frac{D}{Dt}(u^2+v^2)\right]\Delta x\Delta y 1$$

where the total derivative D/Dt for two-dimensional, steady flow considered here is defined as

$$\frac{D}{Dt} \equiv u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}$$

Finally, Previously derived equations are introduced to the energy balance equation and the resulting expression is simplified by combining it with the momentum equation introducing the definition of various stress terms given above.

After quite lengthy manipulations, the energy equation in the rectangular coordinate system for steady, two-dimensional (x,y) flow of an incompressible, constant-property, Newtonian fluid is determined as

$$\rho c_p\left(u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right) = k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)+\mu\Phi$$

where the viscous-energy-dissipation function Φ is defined as

$$\Phi \equiv 2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right]+\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)^2$$

The physical significance of various terms in the above Equation is as follows: The left-hand side represents the net energy transfer into the control volume. The right-hand side terms in parentheses represent conductive heat transfer. And the last term on the right-hand side is the viscous-energy dissipation in the fluid due to internal fluid friction.

For most engineering application, the flow velocities are moderate. Therefore, the viscous-energy dissipation term can be neglected and the axial conduction term $k\frac{\partial^2 T}{\partial x^2}$ is negligible. The energy equation is simplified to

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} \qquad (5\text{-}25)$$

where $\alpha = k/(\rho c_p)$. For the case of no flow (u=v=0), the energy equation simplifies to steady-state heat conduction equation with no heat generation. The energy equation of the boundary layer has a striking similarity with the momentum boundary equation with constant pressure

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \qquad (5\text{-}26)$$

The solution to these two equations would have exactly the same form when $\alpha = \nu$. Thus, we should expect that the relative magnitudes of the thermal diffusivity and kinematic viscosity would have an important influence on convection heat transfer since these magnitudes relate the velocity distribution to the temperature distribution.

The dimensionless parameter, Prandtl Number, Pr

$$\mathrm{Pr} = \frac{\nu}{\alpha} = \frac{\mu c_p}{k}$$

arises from the rate laws incorporated in the governing equations for shear stress and heat flux;

$$\tau = \mu\frac{\partial u}{\partial y} = \rho\nu\left(\frac{\partial u}{\partial y}\right)$$

$$\frac{q}{A} = -k\frac{\partial T}{\partial y} = \rho c_p \alpha\left(\frac{\partial T}{\partial y}\right)$$

Thus, Pr is a measure of the ratio of momentum diffusion through the fluid due to viscosity, to heat diffusion by conduction. Consequently, Pr becomes a measure of the relative size of the two boundary layers-the velocity boundary layer and the thermal boundary layer.

In other words, for fluids having a Prandtl number equal to unity, such as gases, $\delta_t(x) = \delta(x)$.

$\delta_t(x) \gg \delta(x)$ for fluids having $\mathrm{Pr} \ll 1$, such as liquid metals and $\delta_t(x) \ll \delta(x)$ for fluids having $\mathrm{Pr} \gg 1$, such as water and oil.

would have an important influence on convection heat transfer since these magnitudes relate the velocity distribution to the temperature distribution. This is exactly the case, and we shall see the role that these parameters play in the subsequent discussion.

5-6 | THE THERMAL BOUNDARY LAYER

Just as the hydrodynamic boundary layer was defined as that region of the flow where viscous forces are felt, a thermal boundary layer may be defined as that region where temperature gradients are present in the flow. These temperature gradients would result from a heat-exchange process between the fluid and the wall.

Consider the system shown in Figure 5-7. The temperature of the wall is T_w, the temperature of the fluid outside the thermal boundary layer is T_∞, and the thickness of the thermal boundary layer is designated as δ_t. At the wall, the velocity is zero, and the heat transfer into the fluid takes place by conduction. Thus the local heat flux per unit area, q'', is

$$\frac{q}{A} = q'' = -k\left.\frac{\partial T}{\partial y}\right]_{\text{wall}} \tag{5-27}$$

From Newton's law of cooling [Equation (1-8)],

$$q'' = h(T_w - T_\infty) \tag{5-28}$$

where h is the convection heat-transfer coefficient. Combining these equations, we have

$$h = \frac{-k(\partial T/\partial y)_{\text{wall}}}{T_w - T_\infty} \tag{5-29}$$

so that we need only find the temperature gradient at the wall in order to evaluate the heat-transfer coefficient. This means that we must obtain an expression for the temperature distribution. To do this, an approach similar to that used in the momentum analysis of the boundary layer is followed.

The conditions that the temperature distribution must satisfy are

$$T = T_w \qquad \text{at } y = 0 \tag{a}$$

$$\frac{\partial T}{\partial y} = 0 \qquad \text{at } y = \delta_t \tag{b}$$

$$T = T_\infty \qquad \text{at } y = \delta_t \tag{c}$$

Figure 5-7 | Temperature profile in the thermal boundary layer.

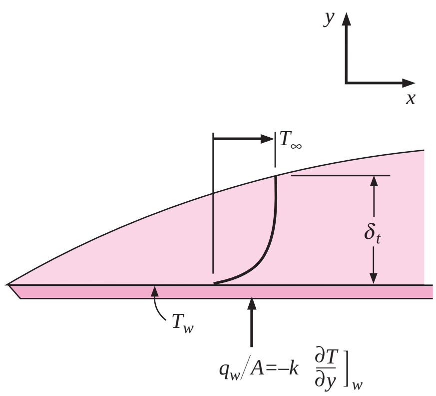
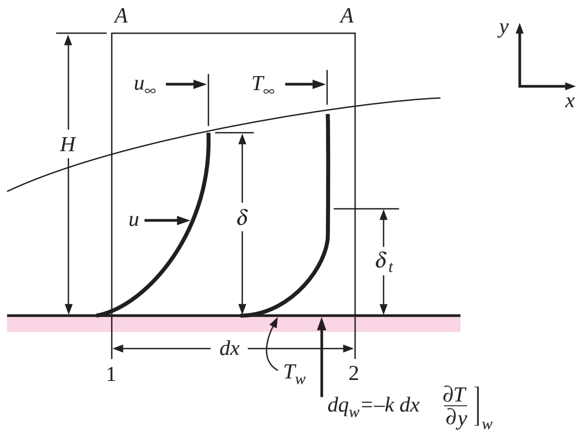


Figure 5-8 | Control volume for integral energy analysis of laminar boundary flow.



and by writing Equation (5-25) at $y = 0$ with no viscous heating we find

$$\frac{\partial^2 T}{\partial y^2} = 0 \qquad \text{at } y = 0 \qquad \textbf{[d]}$$

since the velocities must be zero at the wall.

Conditions (a) to (d) may be fitted to a cubic polynomial as in the case of the velocity profile, so that

$$\frac{\theta}{\theta_\infty} = \frac{T - T_w}{T_\infty - T_w} = \frac{3}{2}\frac{y}{\delta_t} - \frac{1}{2}\left(\frac{y}{\delta_t}\right)^3 \qquad \textbf{[5-30]}$$

where $\theta = T - T_w$. There now remains the problem of finding an expression for δ_t, the thermal-boundary-layer thickness. This may be obtained by an integral analysis of the energy equation for the boundary layer.

Consider the control volume bounded by the planes 1, 2, A-A, and the wall as shown in Figure 5-8. It is assumed that the thermal boundary layer is thinner than the hydrodynamic boundary layer, as shown. The wall temperature is T_w, the free-stream temperature is T_∞, and the heat given up to the fluid over the length dx is dq_w. We wish to make the energy balance

$$\begin{aligned}&\text{Energy convected in} + \text{viscous work within element}\\ &\qquad\qquad + \text{heat transfer at wall} = \text{energy convected out}\end{aligned} \qquad \textbf{[5-31]}$$

The energy convected in through plane 1 is

$$\rho c_p \int_0^H uT\,dy$$

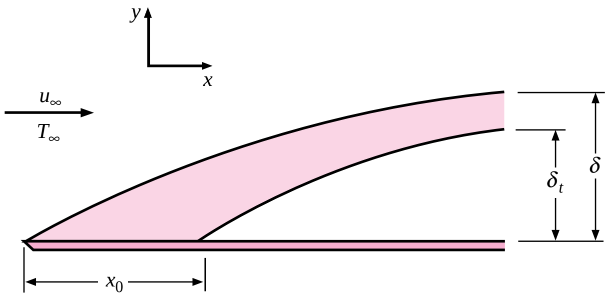
and the energy convected out through plane 2 is

$$\rho c_p \left(\int_0^H uT\,dy\right) + \frac{d}{dx}\left(\rho c_p \int_0^H uT\,dy\right) dx$$

The mass flow through plane A-A is

$$\frac{d}{dx}\left(\int_0^H \rho u\,dy\right) dx$$

Figure 5-9 | Hydrodynamic and thermal boundary layers on a flat plate. Heating starts at $x = x_0$.



and this carries with it an energy equal to

$$c_p T_\infty \frac{d}{dx}\left(\int_0^H \rho u\,dy\right) dx$$

The net viscous work done within the element is

$$\mu\left[\int_0^H \left(\frac{du}{dy}\right)^2 dy\right] dx$$

and the heat transfer at the wall is

$$dq_w = -k\,dx\,\frac{\partial T}{\partial y}\bigg]_w$$

Combining these energy quantities according to Equation (5-31) and collecting terms gives

$$\frac{d}{dx}\left[\int_0^H (T_\infty - T)u\,dy\right] + \frac{\mu}{\rho c_p}\left[\int_0^H \left(\frac{du}{dy}\right)^2 dy\right] = \alpha\,\frac{\partial T}{dy}\bigg]_w \qquad \textbf{[5-32]}$$

This is the integral energy equation of the boundary layer for constant properties and constant free-stream temperature T_∞.

To calculate the heat transfer at the wall, we need to derive an expression for the thermal-boundary-layer thickness that may be used in conjunction with Equations (5-29) and (5-30) to determine the heat-transfer coefficient. For now, we neglect the viscous-dissipation term; this term is very small unless the velocity of the flow field becomes very large. And the calculation of high-velocity heat transfer will be considered later.

The plate under consideration need not be heated over its entire length. The situation that we shall analyze is shown in Figure 5-9, where the hydrodynamic boundary layer develops from the leading edge of the plate, while heating does not begin until $x = x_0$.

Inserting the temperature distribution Equation (5-30) and the velocity distribution Equation (5-19) into Equation (5-32) and neglecting the viscous-dissipation term, gives

$$\begin{aligned}\frac{d}{dx}\left[\int_0^H (T_\infty - T)u\,dy\right] &= \frac{d}{dx}\left[\int_0^H (\theta_\infty - \theta)u\,dy\right]\\ &= \theta_\infty u_\infty \frac{d}{dx}\left\{\int_0^H \left[1 - \frac{3}{2}\frac{y}{\delta_t} + \frac{1}{2}\left(\frac{y}{\delta_t}\right)^3\right]\left[\frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right] dy\right\}\\ &= \alpha\,\frac{\partial T}{\partial y}\bigg]_{y=0} = \frac{3\alpha\theta_\infty}{2\delta_t}\end{aligned}$$

Let us assume that the thermal boundary layer is thinner than the hydrodynamic boundary layer. Then we only need to carry out the integration to $y = \delta_t$ since the integrand is zero for $y > \delta_t$. Performing the necessary algebraic manipulation, carrying out the integration, and making the substitution $\zeta = \delta_t/\delta$ yields

$$\theta_\infty u_\infty \frac{d}{dx}\left[\delta\left(\frac{3}{20}\zeta^2 - \frac{3}{280}\zeta^4\right)\right] = \frac{3}{2}\frac{\alpha\theta_\infty}{\delta\zeta} \tag{5-33}$$

Because $\delta_t < \delta$, $\zeta < 1$, and the term involving ζ^4 is small compared with the ζ^2 term, we neglect the ζ^4 term and write

$$\frac{3}{20}\theta_\infty u_\infty \frac{d}{dx}(\delta\zeta^2) = \frac{3}{2}\frac{\alpha\theta_\infty}{\zeta\delta} \tag{5-34}$$

Performing the differentiation gives

$$\frac{1}{10}u_\infty\left(2\delta\zeta\frac{d\zeta}{dx} + \zeta^2\frac{d\delta}{dx}\right) = \frac{\alpha}{\delta\zeta}$$

or

$$\frac{1}{10}u_\infty\left(2\delta^2\zeta^2\frac{d\zeta}{dx} + \zeta^3\delta\frac{d\delta}{dx}\right) = \alpha$$

But

$$\delta\, d\delta = \frac{140}{13}\frac{\nu}{u_\infty}\, dx$$

and

$$\delta^2 = \frac{280}{13}\frac{\nu x}{u_\infty}$$

so that we have

$$\zeta^3 + 4x\zeta^2\frac{d\zeta}{dx} = \frac{13}{14}\frac{\alpha}{\nu} \tag{5-35}$$

Noting that

$$\zeta^2\frac{d\zeta}{dx} = \frac{1}{3}\frac{d}{dx}\zeta^3$$

we see that Equation (5-35) is a linear differential equation of the first order in ζ^3, and the solution is

$$\zeta^3 = Cx^{-3/4} + \frac{13}{14}\frac{\alpha}{\nu}$$

When the boundary condition

$$\begin{aligned} \delta_t &= 0 \qquad \text{at } x = x_0 \\ \zeta &= 0 \qquad \text{at } x = x_0 \end{aligned}$$

is applied, the final solution becomes

$$\zeta = \frac{\delta_t}{\delta} = \frac{1}{1.026}\mathrm{Pr}^{-1/3}\left[1 - \left(\frac{x_0}{x}\right)^{3/4}\right]^{1/3} \tag{5-36}$$

where

$$\mathrm{Pr} = \frac{\nu}{\alpha} \tag{5-37}$$

has been introduced. The ratio ν/α is called the Prandtl number after Ludwig Prandtl, the German scientist who introduced the concepts of boundary-layer theory.

When the plate is heated over the entire length, $x_0 = 0$, and

$$\frac{\delta_t}{\delta} = \zeta = \frac{1}{1.026} \mathrm{Pr}^{-1/3} \qquad \text{[5-38]}$$

In the foregoing analysis the assumption was made that $\zeta < 1$. This assumption is satisfactory for fluids having Prandtl numbers greater than about 0.7. Fortunately, most gases and liquids fall within this category. Liquid metals are a notable exception, however, since they have Prandtl numbers of the order of 0.01.

The Prandtl number ν/α has been found to be the parameter that relates the relative thicknesses of the hydrodynamic and thermal boundary layers. The kinematic viscosity of a fluid conveys information about the rate at which momentum may diffuse through the fluid because of molecular motion. The thermal diffusivity tells us the same thing in regard to the diffusion of heat in the fluid. Thus the ratio of these two quantities should express the relative magnitudes of diffusion of momentum and heat in the fluid. But these diffusion rates are precisely the quantities that determine how thick the boundary layers will be for a given external flow field; large diffusivities mean that the viscous or temperature influence is felt farther out in the flow field. The Prandtl number is thus the connecting link between the velocity field and the temperature field.

The Prandtl number is dimensionless when a consistent set of units is used:

$$\mathrm{Pr} = \frac{\nu}{\alpha} = \frac{\mu/\rho}{k/\rho c_p} = \frac{c_p \mu}{k} \qquad \text{[5-39]}$$

In the SI system a typical set of units for the parameters would be μ in kilograms per second per meter, c_p in kilojoules per kilogram per Celsius degree, and k in kilowatts per meter per Celsius degree. In the English system one would typically employ μ in pound mass per hour per foot, c_p in Btu per pound mass per Fahrenheit degree, and k in Btu per hour per foot per Fahrenheit degree.

Returning now to the analysis, we have

$$h = \frac{-k(\partial T/\partial y)_w}{T_w - T_\infty} = \frac{3}{2}\frac{k}{\delta_t} = \frac{3}{2}\frac{k}{\zeta\delta} \qquad \text{[5-40]}$$

Substituting for the hydrodynamic-boundary-layer thickness from Equation (5-21) and using Equation (5-36) gives

$$h_x = 0.332k\,\mathrm{Pr}^{1/3}\left(\frac{u_\infty}{\nu x}\right)^{1/2}\left[1 - \left(\frac{x_0}{x}\right)^{3/4}\right]^{-1/3} \qquad \text{[5-41]}$$

The equation may be nondimensionalized by multiplying both sides by x/k, producing the dimensionless group on the left side,

$$\mathrm{Nu}_x = \frac{h_x x}{k} \qquad \text{[5-42]}$$

called the Nusselt number after Wilhelm Nusselt, who made significant contributions to the theory of convection heat transfer. Finally,

$$\mathrm{Nu}_x = 0.332\mathrm{Pr}^{1/3}\,\mathrm{Re}_x^{1/2}\left[1 - \left(\frac{x_0}{x}\right)^{3/4}\right]^{-1/3} \qquad \text{[5-43]}$$

or, for the plate heated over its entire length, $x_0 = 0$ and

$$\mathrm{Nu}_x = 0.332\mathrm{Pr}^{1/3}\,\mathrm{Re}_x^{1/2} \qquad \text{[5-44]}$$

Equations (5-41), (5-43), and (5-44) express the local values of the heat-transfer coefficient in terms of the distance from the leading edge of the plate and the fluid properties. For the case where $x_0 = 0$ the average heat-transfer coefficient and Nusselt number may be obtained by integrating over the length of the plate:

$$\bar{h} = \frac{\int_0^L h_x \, dx}{\int_0^L dx} = 2h_{x=L} \tag{5-45a}$$

For a plate where heating starts at $x = x_0$, it can be shown that the average heat transfer coefficient can be expressed as

$$\frac{\bar{h}_{x_0-L}}{h_{x=L}} = 2L \frac{1 - (x_0/L)^{3/4}}{L - x_0} \tag{5-45b}$$

In this case, the total heat transfer for the plate would be

$$q_{\text{total}} = \bar{h}_{x_0-L}(L - x_0)(T_w - T_\infty)$$

assuming the heated section is at the constant temperature T_w. For the plate heated over the entire length,

$$\overline{\text{Nu}}_L = \frac{\bar{h}L}{k} = 2\,\text{Nu}_{x=L} \tag{5-46a}$$

or

$$\overline{\text{Nu}}_L = \frac{\bar{h}L}{k} = 0.664\,\text{Re}_L^{1/2}\text{Pr}^{1/3} \tag{5-46b}$$

where

$$\text{Re}_L = \frac{\rho u_\infty L}{\mu}$$

The reader should carry out the integrations to verify these results.

The foregoing analysis was based on the assumption that the fluid properties were constant throughout the flow. When there is an appreciable variation between wall and free-stream conditions, it is recommended that the properties be evaluated at the so-called *film temperature* T_f, defined as the arithmetic mean between the wall and free-stream temperature,

$$T_f = \frac{T_w + T_\infty}{2} \tag{5-47}$$

An exact solution to the energy equation is given in Appendix B. The results of the exact analysis are the same as those of the approximate analysis given above.

Constant Heat Flux

The above analysis has considered the laminar heat transfer from an isothermal surface. In many practical problems the surface *heat flux* is essentially constant, and the objective is to find the distribution of the plate-surface temperature for given fluid-flow conditions. For the constant-heat-flux case it can be shown that the local Nusselt number is given by

$$\text{Nu}_x = \frac{hx}{k} = 0.453\,\text{Re}_x^{1/2}\,\text{Pr}^{1/3} \tag{5-48}$$

which may be expressed in terms of the wall heat flux and temperature difference as

$$\text{Nu}_x = \frac{q_w x}{k(T_w - T_\infty)} \tag{5-49}$$

The average temperature difference along the plate, for the constant-heat-flux condition, may be obtained by performing the integration

$$\overline{T_w - T_\infty} = \frac{1}{L}\int_0^L (T_w - T_\infty)\,dx = \frac{1}{L}\int_0^L \frac{q_w x}{k\,\mathrm{Nu}_x}\,dx$$
$$= \frac{q_w L/k}{0.6795\,\mathrm{Re}_L^{1/2}\,\mathrm{Pr}^{1/3}} \qquad \textbf{[5-50]}$$

or

$$q_w = \tfrac{3}{2} h_{x=L}(\overline{T_w - T_\infty})$$

In these equations q_w is the heat flux per unit area and will have the units of watts per square meter (W/m^2) in SI units or British thermal units per hour per square foot (Btu/h $\cdot$ ft^2) in the English system. Note that the heat flux $q_w = q/A$ is assumed constant over the entire plate surface.

Other Relations

Equation (5-44) is applicable to fluids having Prandtl numbers between about 0.6 and 50. It would not apply to fluids with very low Prandtl numbers like liquid metals or to high-Prandtl-number fluids like heavy oils or silicones. For a very wide range of Prandtl numbers, Churchill and Ozoe [9] have correlated a large amount of data to give the following relation for laminar flow on an isothermal flat plate:

$$\mathrm{Nu}_x = \frac{0.3387\,\mathrm{Re}_x^{1/2}\,\mathrm{Pr}^{1/3}}{\left[1 + \left(\dfrac{0.0468}{\mathrm{Pr}}\right)^{2/3}\right]^{1/4}} \qquad \text{for } \mathrm{Re}_x\,\mathrm{Pr} > 100 \qquad \textbf{[5-51]}$$

For the constant-heat-flux case, 0.3387 is changed to 0.4637 and 0.0468 is changed to 0.0207. Properties are still evaluated at the film temperature.

Isothermal Flat Plate Heated Over Entire Length — EXAMPLE 5-4

For the flow system in Example 5-3 assume that the plate is heated over its entire length to a temperature of 60°C. Calculate the heat transferred in (*a*) the first 20 cm of the plate and (*b*) the first 40 cm of the plate.

■ Solution

The total heat transfer over a certain length of the plate is desired; so we wish to calculate average heat-transfer coefficients. For this purpose we use Equations (5-44) and (5-45), evaluating the properties at the film temperature:

$$T_f = \frac{27 + 60}{2} = 43.5^\circ\mathrm{C} = 316.5\ \mathrm{K} \quad [110.3^\circ\mathrm{F}]$$

From Appendix A the properties are

$$\nu = 17.36 \times 10^{-6}\ \mathrm{m^2/s} \quad [1.87 \times 10^{-4}\ \mathrm{ft^2/s}]$$
$$k = 0.02749\ \mathrm{W/m\cdot{}^\circ C} \quad [0.0159\ \mathrm{Btu/h\cdot ft\cdot{}^\circ F}]$$
$$\mathrm{Pr} = 0.7$$
$$c_p = 1.006\ \mathrm{kJ/kg\cdot{}^\circ C} \quad [0.24\ \mathrm{Btu/lb_m\cdot{}^\circ F}]$$

At $x = 20$ cm

$$\mathrm{Re}_x = \frac{u_\infty x}{\nu} = \frac{(2)(0.2)}{17.36 \times 10^{-6}} = 23{,}041$$

$$\mathrm{Nu}_x = \frac{h_x x}{k} = 0.332\mathrm{Re}_x^{1/2}\mathrm{Pr}^{1/3}$$

$$= (0.332)(23{,}041)^{1/2}(0.7)^{1/3} = 44.74$$

$$h_x = \mathrm{Nu}_x\left(\frac{k}{x}\right) = \frac{(44.74)(0.02749)}{0.2}$$

$$= 6.15\ \mathrm{W/m^2\cdot{}^\circ C} \quad [1.083\ \mathrm{Btu/h\cdot ft^2\cdot{}^\circ F}]$$

The average value of the heat-transfer coefficient is twice this value, or

$$\bar{h} = (2)(6.15) = 12.3\ \mathrm{W/m^2\cdot{}^\circ C} \quad [2.17\ \mathrm{Btu/h\cdot ft^2\cdot{}^\circ F}]$$

The heat flow is

$$q = \bar{h}A(T_w - T_\infty)$$

If we assume unit depth in the z direction,

$$q = (12.3)(0.2)(60 - 27) = 81.18\ \mathrm{W} \quad [277\ \mathrm{Btu/h}]$$

At $x = 40$ cm

$$\mathrm{Re}_x = \frac{u_\infty x}{\nu} = \frac{(2)(0.4)}{17.36 \times 10^{-6}} = 46{,}082$$

$$\mathrm{Nu}_x = (0.332)(46{,}082)^{1/2}(0.7)^{1/3} = 63.28$$

$$h_x = \frac{(63.28)(0.02749)}{0.4} = 4.349\ \mathrm{W/m^2\cdot{}^\circ C}$$

$$\bar{h} = (2)(4.349) = 8.698\ \mathrm{W/m^2\cdot{}^\circ C} \quad [1.53\ \mathrm{Btu/h\cdot ft^2\cdot{}^\circ F}]$$

$$q = (8.698)(0.4)(60 - 27) = 114.8\ \mathrm{W} \quad [392\ \mathrm{Btu/h}]$$

EXAMPLE 5-5 Flat Plate with Constant Heat Flux

A 1.0-kW heater is constructed of a glass plate with an electrically conducting film that produces a constant heat flux. The plate is 60 cm by 60 cm and placed in an airstream at 27°C, 1 atm with $u_\infty = 5$ m/s. Calculate the average temperature difference along the plate and the temperature difference at the trailing edge.

■ Solution

Properties should be evaluated at the film temperature, but we do not know the plate temperature. So for an initial calculation, we take the properties at the free-stream conditions of

$$T_\infty = 27^\circ\mathrm{C} = 300\ \mathrm{K}$$

$$\nu = 15.69 \times 10^{-6}\ \mathrm{m^2/s} \qquad \mathrm{Pr} = 0.708 \qquad k = 0.02624\ \mathrm{W/m\cdot{}^\circ C}$$

$$\mathrm{Re}_L = \frac{(0.6)(5)}{15.69 \times 10^{-6}} = 1.91 \times 10^5$$

From Equation (5-50) the average temperature difference is

$$\overline{T_w - T_\infty} = \frac{[1000/(0.6)^2](0.6)/0.02624}{0.6795(1.91 \times 10^5)^{1/2}(0.708)^{1/3}} = 240^\circ\mathrm{C}$$

Now, we go back and evaluate properties at

$$T_f = \frac{240 + 27 + 27}{2} = 147°\text{C} = 420 \text{ K}$$

and obtain

$$\nu = 28.22 \times 10^{-6} \text{ m}^2/\text{s} \qquad \text{Pr} = 0.687 \qquad k = 0.035 \text{ W/m} \cdot °\text{C}$$

$$\text{Re}_L = \frac{(0.6)(5)}{28.22 \times 10^{-6}} = 1.06 \times 10^5$$

$$\overline{T_w - T_\infty} = \frac{[1000/(0.6)^2](0.6)/0.035}{0.6795(1.06 \times 10^5)^{1/2}(0.687)^{1/3}} = 243°\text{C}$$

At the end of the plate ($x = L = 0.6$ m) the temperature difference is obtained from Equations (5-48) and (5-50) with the constant 0.453 to give

$$(T_w - T_\infty)_{x=L} = \frac{(243.6)(0.6795)}{0.453} = 365.4°\text{C}$$

An alternate solution would be to base the Nusselt number on Equation (5-51).

Plate with Unheated Starting Length — EXAMPLE 5-6

Air at 1 atm and 300 K flows across a 20-cm-square plate at a free-stream velocity of 20 m/s. The last half of the plate is heated to a constant temperature of 350 K. Calculate the heat lost by the plate.

■ Solution

First we evaluate the air properties at the film temperature

$$T_f = (T_w + T_\infty)/2 = 325 \text{ K}$$

and obtain

$$\nu = 18.23 \times 10^{-6} \text{m}^2/\text{s} \qquad k = 0.02814 \text{ W/m} \cdot °\text{C} \qquad \text{Pr} = 0.7$$

At the trailing edge of the plate the Reynolds number is

$$\text{Re}_L = u_\infty L/\nu = (20)(0.2)/18.23 \times 10^{-6} = 2.194 \times 10^5$$

or, laminar flow over the length of the plate.

Heating does not start until the last half of the plate, or at a position $x_0 = 0.1$ m. The local heat-transfer coefficient for this condition is given by Equation (5-41):

$$h_x = 0.332k\, \text{Pr}^{1/3}(u_\infty/\nu x)^{1/2}[1 - (x_0/x)^{0.75}]^{-1/3} \qquad \textbf{[a]}$$

Inserting the property values along with $x_0 = 0.1$ gives

$$h_x = 8.6883x^{-1/2}(1 - 0.17783x^{-0.75})^{-1/3} \qquad \textbf{[b]}$$

The plate is 0.2 m wide so the heat transfer is obtained by integrating over the heated length $x_0 < x < L$

$$q = (0.2)(T_w - T_\infty)\int_{x_0=0.1}^{L=0.2} h_x dx \qquad \textbf{[c]}$$

Inserting Equation (b) in Equation (c) and performing the numerical integration gives

$$q = (0.2)(8.6883)(0.4845)(350 - 300) = 421 \text{ W} \qquad \textbf{[}d\textbf{]}$$

The average value of the heat-transfer coefficient *over the heated length* is given by

$$h = q/(T_w - T_\infty)(L - x_0)W = 421/(350 - 300)(0.2 - 0.1)(0.2) = 421 \text{ W/m}^2 \cdot {}^\circ\text{C}$$

where W is the width of the plate.

An easier calculation can be made by applying Equation (5-45b) to determine the average heat transfer coefficient over the heated portion of the plate. The result is

$$h = 425.66 \text{ W/m}^2 \cdot {}^\circ\text{C} \quad \text{and} \quad q = 425.66 \text{ W}$$

which indicates, of course, only a small error in the numerical integration.

EXAMPLE 5-7 Oil Flow Over Heated Flat Plate

Engine oil at 20°C is forced over a 20-cm-square plate at a velocity of 1.2 m/s. The plate is heated to a uniform temperature of 60°C. Calculate the heat lost by the plate.

■ Solution

We first evaluate the film temperature:

$$T_f = \frac{20 + 60}{2} = 40^\circ\text{C}$$

The properties of engine oil are

$$\rho = 876 \text{ kg/m}^3 \qquad \nu = 0.00024 \text{ m}^2/\text{s}$$
$$k = 0.144 \text{ W/m} \cdot {}^\circ\text{C} \qquad \text{Pr} = 2870$$

The Reynolds number is

$$\text{Re} = \frac{u_\infty L}{\nu} = \frac{(1.2)(0.2)}{0.00024} = 1000$$

Because the Prandtl number is so large we will employ Equation (5-51) for the solution. We see that h_x varies with x in the same fashion as in Equation (5-44), that is, $h_x \propto x^{-1/2}$, so that we get the same solution as in Equation (5-45) for the average heat-transfer coefficient. Evaluating Equation (5-51) at $x = 0.2$ gives

$$\text{Nu}_x = \frac{(0.3387)(1000)^{1/2}(2870)^{1/3}}{\left[1 + \left(\dfrac{0.0468}{2870}\right)^{2/3}\right]^{1/4}} = 152.2$$

and

$$h_x = \frac{(152.2)(0.144)}{0.2} = 109.6 \text{ W/m}^2 \cdot {}^\circ\text{C}$$

The average value of the convection coefficient is

$$h = (2)(109.6) = 219.2 \text{ W/m}^2 \cdot {}^\circ\text{C}$$

so that the total heat transfer is

$$q = hA(T_w - T_\infty) = (219.2)(0.2)^2(60 - 20) = 350.6 \text{ W}$$

5-7 | THE RELATION BETWEEN FLUID FRICTION AND HEAT TRANSFER

We have already seen that the temperature and flow fields are related. Now we seek an expression whereby the frictional resistance may be directly related to heat transfer.

The shear stress at the wall may be expressed in terms of a friction coefficient C_f:

$$\tau_w = C_f \frac{\rho u_\infty^2}{2} \qquad \textbf{[5-52]}$$

Equation (5-52) is the defining equation for the friction coefficient. The shear stress may also be calculated from the relation

$$\tau_w = \mu \left.\frac{\partial u}{\partial y}\right]_w$$

Using the velocity distribution given by Equation (5-19), we have

$$\tau_w = \frac{3}{2} \frac{\mu u_\infty}{\delta}$$

and making use of the relation for the boundary-layer thickness gives

$$\tau_w = \frac{3}{2} \frac{\mu u_\infty}{4.64} \left(\frac{u_\infty}{\nu x}\right)^{1/2} \qquad \textbf{[5-53]}$$

Combining Equations (5-52) and (5-53) leads to

$$\frac{C_{fx}}{2} = \frac{3}{2} \frac{\mu u_\infty}{4.64} \left(\frac{u_\infty}{\nu x}\right)^{1/2} \frac{1}{\rho u_\infty^2} = 0.323 \, \mathrm{Re}_x^{-1/2} \qquad \textbf{[5-54]}$$

The exact solution of the boundary-layer equations yields

$$\frac{C_{fx}}{2} = 0.332 \, \mathrm{Re}_x^{-1/2} \qquad \textbf{[5-54a]}$$

Equation (5-44) may be rewritten in the following form:

$$\frac{\mathrm{Nu}_x}{\mathrm{Re}_x \, \mathrm{Pr}} = \frac{h_x}{\rho c_p u_\infty} = 0.332 \, \mathrm{Pr}^{-2/3} \, \mathrm{Re}_x^{-1/2}$$

The group on the left is called the Stanton number,

$$\mathrm{St}_x = \frac{h_x}{\rho c_p u_\infty}$$

so that

$$\mathrm{St}_x \, \mathrm{Pr}^{2/3} = 0.332 \, \mathrm{Re}_x^{1/2} \qquad \textbf{[5-55]}$$

Upon comparing Equations (5-54) and (5-55), we note that the right sides are alike except for a difference of about 3 percent in the constant, which is the result of the approximate nature of the integral boundary-layer analysis. We recognize this approximation and write

$$\mathrm{St}_x \mathrm{Pr}^{2/3} = \frac{C_{fx}}{2} \tag{5-56}$$

Equation (5-56), called the *Reynolds-Colburn analogy*, expresses the relation between fluid friction and heat transfer for laminar flow on a flat plate. The heat-transfer coefficient thus could be determined by making measurements of the frictional drag on a plate under conditions in which no heat transfer is involved.

It turns out that Equation (5-56) can also be applied to turbulent flow over a flat plate and in a modified way to turbulent flow in a tube. It does not apply to laminar tube flow. In general, a more rigorous treatment of the governing equations is necessary when embarking on new applications of the heat-transfer–fluid-friction analogy, and the results do not always take the simple form of Equation (5-56). The interested reader may consult the references at the end of the chapter for more information on this important subject. At this point, the simple analogy developed above has served to amplify our understanding of the physical processes in convection and to reinforce the notion that heat-transfer and viscous-transport processes are related at both the microscopic and macroscopic levels.

EXAMPLE 5-8 Drag Force on a Flat Plate

For the flow system in Example 5-4 compute the drag force exerted on the first 40 cm of the plate using the analogy between fluid friction and heat transfer.

Solution

We use Equation (5-56) to compute the friction coefficient and then calculate the drag force. An average friction coefficient is desired, so

$$\overline{\mathrm{St}}\, \mathrm{Pr}^{2/3} = \frac{\overline{C}_f}{2} \tag{a}$$

The density at 316.5 K is

$$\rho = \frac{p}{RT} = \frac{1.0132 \times 10^5}{(287)(316.5)} = 1.115 \text{ kg/m}^3$$

For the 40-cm length

$$\overline{\mathrm{St}} = \frac{\bar{h}}{\rho c_p u_\infty} = \frac{8.698}{(1.115)(1006)(2)} = 3.88 \times 10^{-3}$$

Then from Equation (*a*)

$$\frac{\overline{C}_f}{2} = (3.88 \times 10^{-3})(0.7)^{2/3} = 3.06 \times 10^{-3}$$

The average shear stress at the wall is computed from Equation (5-52):

$$\begin{aligned}\bar{\tau}_w &= \overline{C}_f \rho \frac{u_\infty^2}{2} \\ &= (3.06 \times 10^{-3})(1.115)(2)^2 \\ &= 0.0136 \text{ N/m}^2\end{aligned}$$

The drag force is the product of this shear stress and the area,

$$D = (0.0136)(0.4) = 5.44 \text{ mN} \quad [1.23 \times 10^{-3} \text{ lb}_f]$$

5-8 | TURBULENT-BOUNDARY-LAYER HEAT TRANSFER

Consider a portion of a turbulent boundary layer as shown in Figure 5-10. A very thin region near the plate surface has a laminar character, and the viscous action and heat transfer take place under circumstances like those in laminar flow. Farther out, at larger y distances from the plate, some turbulent action is experienced, but the molecular viscous action and heat conduction are still important. This region is called the *buffer layer.* Still farther out, the flow is fully turbulent, and the main momentum- and heat-exchange mechanism is one involving macroscopic lumps of fluid moving about in the flow. In this fully turbulent region we speak of *eddy viscosity* and *eddy thermal conductivity.* These eddy properties may be 10 to 20 times as large as the molecular values.

The physical mechanism of heat transfer in turbulent flow is quite similar to that in laminar flow; the primary difference is that one must deal with the eddy properties instead of the ordinary thermal conductivity and viscosity. The main difficulty in an analytical treatment is that these eddy properties vary across the boundary layer, and the specific variation can be determined only from experimental data. This is an important point. All analyses of turbulent flow must eventually rely on experimental data because there is no completely adequate theory to predict turbulent-flow behavior.

If one observes the instantaneous macroscopic velocity in a turbulent-flow system, as measured with a laser anemometer or other sensitive device, significant fluctuations about the mean flow velocity are observed as indicated in Figure 5-11, where $\overline{u}$ is designated as the mean velocity and u' is the *fluctuation* from the mean. The instantaneous velocity is therefore

$$u = \overline{u} + u' \qquad \textbf{[5-57]}$$

The mean value of the fluctuation u' must be zero over an extended period for steady flow conditions. There are also fluctuations in the y component of velocity, so we would write

$$v = \overline{v} + v' \qquad \textbf{[5-58]}$$

The fluctuations give rise to a turbulent-shear stress that may be analyzed by referring to Figure 5-12.

For a unit area of the plane *P-P*, the instantaneous turbulent mass-transport rate across the plane is $\rho v'$. Associated with this mass transport is a change in the x component of

Figure 5-10 | Velocity profile in turbulent boundary layer on a flat plate.

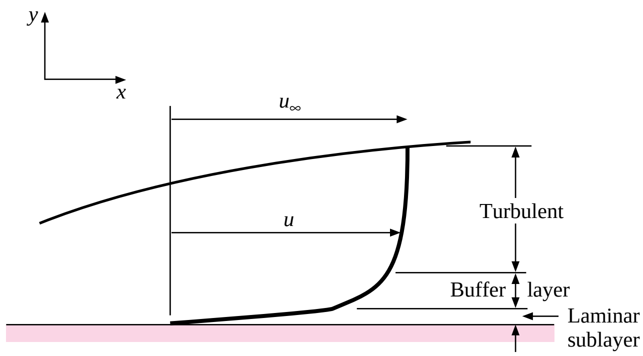


Figure 5-11 | Turbulent fluctuations with time.

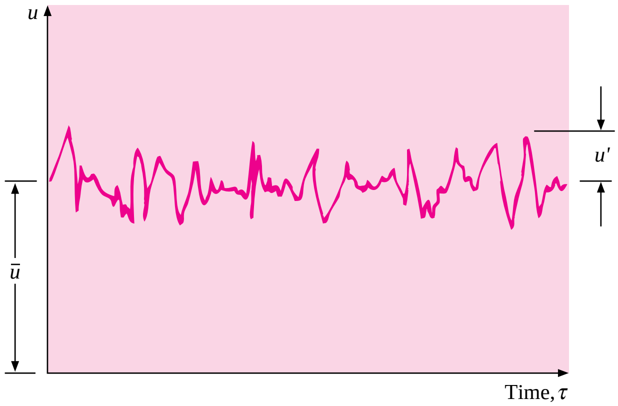
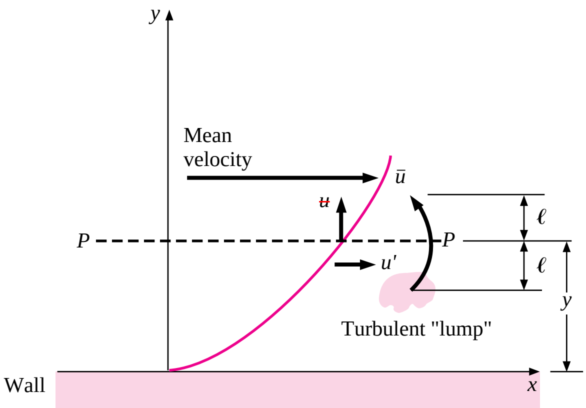


Figure 5-12 | Turbulent shear stress and mixing length.



velocity u'. The net momentum flux per unit area, in the x direction, represents the turbulent-shear stress at the plane P-P, or $\rho v' u'$. When a turbulent lump moves upward ($v' > 0$), it enters a region of higher $\overline{u}$ and is therefore likely to effect a slowing-down fluctuation in u', that is, $u' < 0$. A similar argument can be made for $v' < 0$, so that the average turbulent-shear stress will be given as

$$\tau_t = -\rho \overline{v' u'} \quad \textbf{[5-59]}$$

We must note that even though $\overline{v'} = \overline{u'} = 0$, the average of the *fluctuation product* $\overline{u'v'}$ is *not* zero.

Eddy Viscosity and the Mixing Length

Let us define an eddy viscosity or eddy diffusivity for momentum ϵ_M such that

$$\tau_t = -\rho \overline{v' u'} = \rho \epsilon_M \frac{du}{dy} \quad \textbf{[5-60]}$$

We have already likened the macroscopic transport of heat and momentum in turbulent flow to their molecular counterparts in laminar flow, so the definition in Equation (5-60) is a natural consequence of this analogy. To analyze molecular-transport problems one normally introduces the concept of *mean free path,* or the average distance a particle travels between collisions. Prandtl introduced a similar concept for describing turbulent-flow phenomena. The *Prandtl mixing length* is the distance traveled, on the average, by the turbulent lumps of fluid in a direction normal to the mean flow.

Let us imagine a turbulent lump that is located a distance ℓ above or below the plane *P-P*, as shown in Figure 5-12. These lumps of fluid move back and forth across the plane and give rise to the eddy or turbulent-shear-stress effect. At $y+\ell$ the velocity would be approximately

$$u(y+\ell) \approx u(y) + \ell \frac{\partial u}{\partial y}$$

while at $y-\ell$,

$$u(y-\ell) \approx u(y) - \ell \frac{\partial u}{\partial y}$$

Prandtl postulated that the turbulent fluctuation u' is proportional to the mean of the above two quantities, or

$$u' \approx \ell \frac{\partial u}{\partial y} \tag{5-61}$$

The distance ℓ is called the Prandtl mixing length. Prandtl also postulated that v' would be of the same order of magnitude as u' so that the turbulent-shear stress of Equation (5-60) could be written

$$\tau_t = -\overline{\rho u' v'} = \rho \ell^2 \left(\frac{\partial u}{\partial y}\right)^2 = \rho \epsilon_M \frac{\partial u}{\partial y} \tag{5-62}$$

The eddy viscosity ϵ_M thus becomes

$$\epsilon_M = \ell^2 \frac{\partial u}{\partial y} \tag{5-63}$$

We have already noted that the eddy properties, and hence the mixing length, vary markedly through the boundary layer. Many analysis techniques have been applied over the years to take this variation into account. Prandtl's hypothesis was that the mixing length is proportional to distance from the wall, or

$$\ell = Ky \tag{5-64}$$

where K is the proportionality constant. The additional assumption was made that in the near-wall region the shear stress is approximately constant so that $\tau_t \approx \tau_w$. When this assumption is used along with Equation (5-64), Equation (5-62) yields

$$\tau_w = \rho K^2 y^2 \left(\frac{\partial u}{\partial y}\right)^2$$

Taking the square root and integrating with respect to y gives

$$u = \frac{1}{K}\sqrt{\frac{\tau_w}{\rho}} \ln y + C \tag{5-65}$$

where C is the constant of integration. Equation (5-65) matches very well with experimental data except in the region very close to the wall, where the laminar sublayer is present. In the sublayer the velocity distribution is essentially linear.

Let us now quantify our earlier qualitative description of a turbulent boundary layer by expressing the shear stress as the sum of a molecular and turbulent part:

$$\frac{\tau}{\rho} = (\nu + \epsilon_M)\frac{\partial u}{\partial y} \tag{5-66}$$

The so-called universal velocity profile is obtained by introducing two nondimensional coordinates

$$u^+ = \frac{u}{\sqrt{\tau_w/\rho}} \tag{5-67}$$

$$y^+ = \frac{\sqrt{\tau_w/\rho y}}{\nu} \tag{5-68}$$

Using these parameters and assuming $\tau \approx$ constant, we can rewrite Equation (5-66) as

$$du^+ = \frac{dy^+}{1 + \epsilon_M/\nu} \tag{5-69}$$

In terms of our previous qualitative discussion, the laminar sublayer is the region where $\epsilon_M \sim 0$, the buffer layer has $\epsilon_M \sim \nu$, and the turbulent layer has $\epsilon_M \gg \nu$. Therefore, taking $\epsilon_M = 0$ in Equation (5-69) and integrating yields

$$u^+ = y^+ + c$$

At the wall, $u^+ = 0$ for $y^+ = 0$ so that $c = 0$ and

$$u^+ = y^+ \tag{5-70}$$

is the velocity relation (a linear one) for the laminar sublayer. In the fully turbulent region $\epsilon_M/\nu \gg 1$. From Equation (5-65)

$$\frac{\partial u}{\partial y} = \frac{1}{K}\sqrt{\frac{\tau_w}{\rho}}\,\frac{1}{y}$$

Substituting this relation along with Equation (5-64) into Equation (5-63) gives

$$\epsilon_M = K\sqrt{\frac{\tau_w}{\rho}}\,y$$

or

$$\frac{\epsilon_m}{\nu} = Ky^+ \tag{5-71}$$

Substituting this relation in Equation (5-69) for $\epsilon_M/\nu \gg 1$ and integrating gives

$$u^+ = \frac{1}{K}\ln y^+ + c \tag{5-72}$$

This same *form* of equation will also be obtained for the buffer region. The limits of each region are obtained by comparing the above equations with experimental velocity measurements, with the following generally accepted constants:

$$\begin{aligned}
&\text{Laminar sublayer: } 0 < y^+ < 5 && u^+ = y^+ \\
&\text{Buffer layer: } 5 < y^+ < 30 && u^+ = 5.0\ln y^+ - 3.05 \\
&\text{Turbulent layer: } 30 < y^+ < 400 && u^+ = 2.5\ln y^+ + 5.5
\end{aligned} \tag{5-73}$$

The equation set (5-73) is called the *universal velocity profile* and matches very well with experimental data; however, we should note once again that the constants in the equations must be determined from experimental velocity measurements. The satisfying point is that the simple Prandtl mixing-length model yields an equation form that fits the data so well.

Turbulent heat transfer is analogous to turbulent momentum transfer. The turbulent momentum flux postulated by Equation (5-59) carries with it a turbulent energy fluctuation proportional to the temperature gradient. We thus have, in analogy to Equation (5-62),

$$\left(\frac{q}{A}\right)_{\text{turb}} = -\rho c_p \epsilon_H \frac{\partial T}{\partial y} \qquad \textbf{[5-74]}$$

or, for regions where both molecular and turbulent energy transport are important,

$$\frac{q}{A} = -\rho c_p (\alpha + \epsilon_H) \frac{\partial T}{\partial y} \qquad \textbf{[5-75]}$$

Turbulent Heat Transfer Based on Fluid-Friction Analogy

Various analyses, similar to the one for the universal velocity profile above, have been performed to predict turbulent-boundary-layer heat transfer. The analyses have met with good success, but for our purposes the Colburn analogy between fluid friction and heat transfer is easier to apply and yields results that are in agreement with experiment and of simpler form.

In the turbulent-flow region, where $\epsilon_M \gg \nu$ and $\epsilon_H \gg \alpha$, we define the turbulent Prandtl number as

$$\Pr_t = \frac{\epsilon_M}{\epsilon_H} \qquad \textbf{[5-76]}$$

If we can expect that the eddy momentum and energy transport will both be increased in the same proportion compared with their molecular values, we might anticipate that heat-transfer coefficients can be calculated by Equation (5-56) with the ordinary molecular Prandtl number used in the computation. In the turbulent core of the boundary layer the eddy viscosity may be as high as 100 times the molecular value experienced in the laminar sublayer, and a similar behavior is experienced for the eddy diffusivity for heat compared to the molecular diffusivity. To account for the Prandtl number effect over the entire boundary layer a weighted average is needed, and it turns out that use of $\Pr^{2/3}$ works very well and matches with the laminar heat-transfer–fluid-friction analogy. We thus will base our calculations on this analogy, and we need experimental values for C_f for turbulent boundary layer flows to carry out these computations.

Schlichting [1] has surveyed experimental measurements of friction coefficients for turbulent flow on flat plates. We present the results of that survey so that they may be employed in the calculation of turbulent heat transfer with the fluid-friction–heat-transfer analogy. The *local* skin-friction coefficient is given by

$$C_{fx} = 0.0592\ \mathrm{Re}_x^{-1/5} \qquad \textbf{[5-77]}$$

for Reynolds numbers between 5×10^5 and 10^7. At higher Reynolds numbers from 10^7 to 10^9 the formula of Schultz-Grunow [8] is recommended:

$$C_{fx} = 0.370(\log \mathrm{Re}_x)^{-2.584} \qquad \textbf{[5-78]}$$

The *average-friction coefficient* for a flat plate with a laminar boundary layer up to $\mathrm{Re}_{\mathrm{crit}}$ and turbulent thereafter can be calculated from

$$\overline{C}_f = \frac{0.455}{(\log \mathrm{Re}_L)^{2.584}} - \frac{A}{\mathrm{Re}_L} \qquad \mathrm{Re}_L < 10^9 \tag{5-79}$$

where the constant A depends on $\mathrm{Re}_{\mathrm{crit}}$ in accordance with Table 5-1. A somewhat simpler formula can be obtained for lower Reynolds numbers as

$$\overline{C}_f = \frac{0.074}{\mathrm{Re}_L^{1/5}} - \frac{A}{\mathrm{Re}_L} \qquad \mathrm{Re}_L < 10^7 \tag{5-80}$$

Table 5-1

$\mathrm{Re}_{\mathrm{crit}}$	3×10^5	5×10^5	10^6	3×10^6
A	1055	1742	3340	8940

Equations (5-79) and (5-80) are in agreement within their common range of applicability, and the one to be used in practice will depend on computational convenience.

Applying the fluid-friction analogy St $\mathrm{Pr}^{2/3} = C_f/2$, we obtain the local turbulent heat transfer as:

$$\mathrm{St}_x \, \mathrm{Pr}^{2/3} = 0.0296 \, \mathrm{Re}_x^{-1/5} \qquad 5 \times 10^5 < \mathrm{Re}_x < 10^7 \tag{5-81}$$

or

$$\mathrm{St}_x \, \mathrm{Pr}^{2/3} = 0.185(\log \mathrm{Re}_x)^{-2.584} \qquad 10^7 < \mathrm{Re}_x < 10^9 \tag{5-82}$$

The average heat transfer over the entire laminar-turbulent boundary layer is

$$\overline{\mathrm{St}} \, \mathrm{Pr}^{2/3} = \frac{\overline{C}_f}{2} \tag{5-83}$$

For $\mathrm{Re}_{\mathrm{crit}} = 5 \times 10^5$ and $\mathrm{Re}_L < 10^7$, Equation (5-80) can be used to obtain

$$\overline{\mathrm{St}} \, \mathrm{Pr}^{2/3} = 0.037 \, \mathrm{Re}_L^{-1/5} - 871 \, \mathrm{Re}_L^{-1} \tag{5-84}$$

Recalling that $\overline{\mathrm{St}} = \overline{\mathrm{Nu}}/(\mathrm{Re}_L \, \mathrm{Pr})$, we can rewrite Equation (5-84) as

$$\overline{\mathrm{Nu}}_L = \frac{\bar{h}L}{k} = \mathrm{Pr}^{1/3}(0.037 \, \mathrm{Re}_L^{0.8} - 871) \tag{5-85}$$

The average heat-transfer coefficient can also be obtained by integrating the local values over the entire length of the plate. Thus,

$$h = \frac{1}{L}\left(\int_0^{x_{\mathrm{crit}}} h_{\mathrm{lam}} \, dx + \int_{x_{\mathrm{crit}}}^{L} h_{\mathrm{turb}} \, dx\right)$$

Using Equation (5-55) for the laminar portion, $\mathrm{Re}_{\mathrm{crit}} = 5 \times 10^5$, and Equation (5-81) for the turbulent portion gives the same result as Equation (5-85). For higher Reynolds numbers the friction coefficient from Equation (5-79) may be used, so that

$$\mathrm{Nu}_L = \frac{\bar{h}L}{k} = [0.228\mathrm{Re}_L(\log \mathrm{Re}_L)^{-2.584} - 871]\mathrm{Pr}^{1/3} \qquad \textbf{[5-85a]}$$

for $10^7 < \mathrm{Re}_L < 10^9$ and $\mathrm{Re}_{crit} = 5 \times 10^5$.

The reader should note that if a transition Reynolds number different from 500,000 is chosen, then Equations (5-84) and (5-85) must be changed accordingly. An alternative equation is suggested by Whitaker [10] that may give better results with some liquids because of the viscosity-ratio term:

$$\overline{\mathrm{Nu}}_L = 0.036\,\mathrm{Pr}^{0.43}(\mathrm{Re}_L^{0.8} - 9200)\left(\frac{\mu_\infty}{\mu_w}\right)^{1/4} \qquad \textbf{[5-86]}$$

for

$$0.7 < \mathrm{Pr} < 380$$
$$2 \times 10^5 < \mathrm{Re}_L < 5.5 \times 10^6$$
$$0.26 < \frac{\mu_\infty}{\mu_w} < 3.5$$

All properties except μ_w are evaluated at the free-stream temperature. For gases the viscosity ratio is dropped and the properties are evaluated at the film temperature.

Constant Heat Flux

For constant-wall-heat flux in turbulent flow it is shown in Reference 11 that the local Nusselt number is only about 4 percent higher than for the isothermal surface; that is,

$$\mathrm{Nu}_x = 1.04\,\mathrm{Nu}_x\Big]_{T_w=\mathrm{const}} \qquad \textbf{[5-87]}$$

Some more comprehensive methods of correlating turbulent-boundary-layer heat transfer are given by Churchill [11].

Turbulent Heat Transfer from Isothermal Flat Plate

EXAMPLE 5-9

Air at 20°C and 1 atm flows over a flat plate at 35 m/s. The plate is 75 cm long and is maintained at 60°C. Assuming unit depth in the z direction, calculate the heat transfer from the plate.

■ Solution

We evaluate properties at the film temperature:

$$T_f = \frac{20+60}{2} = 40°\mathrm{C} = 313\ \mathrm{K}$$

$$\rho = \frac{p}{RT} = \frac{1.0132 \times 10^5}{(287)(313)} = 1.128\ \mathrm{kg/m^3}$$

$$\mu = 1.906 \times 10^{-5}\ \mathrm{kg/m \cdot s}$$

$$\mathrm{Pr} = 0.7 \qquad k = 0.02723\ \mathrm{W/m \cdot °C} \qquad c_p = 1.007\ \mathrm{kJ/kg \cdot °C}$$

The Reynolds number is

$$\mathrm{Re}_L = \frac{\rho u_\infty L}{\mu} = \frac{(1.128)(35)(0.75)}{1.906 \times 10^{-5}} = 1.553 \times 10^6$$

and the boundary layer is turbulent because the Reynolds number is greater than 5×10^5. Therefore, we use Equation (5-85) to calculate the average heat transfer over the plate:

$$\overline{\mathrm{Nu}}_L = \frac{\bar{h}L}{k} = \mathrm{Pr}^{1/3}(0.037\,\mathrm{Re}_L^{0.8} - 871)$$

$$= (0.7)^{1/3}[(0.037)(1.553 \times 10^6)^{0.8} - 871] = 2180$$

$$\bar{h} = \overline{\mathrm{Nu}}_L \frac{k}{L} = \frac{(2180)(0.02723)}{0.75} = 79.1\ \mathrm{W/m^2 \cdot {}^\circ C} \quad [13.9\ \mathrm{Btu/h \cdot ft^2 \cdot {}^\circ F}]$$

$$q = \bar{h}A(T_w - T_\infty) = (79.1)(0.75)(60 - 20) = 2373\ \mathrm{W} \quad [8150\ \mathrm{Btu/h}]$$

5-9 | TURBULENT-BOUNDARY-LAYER THICKNESS

A number of experimental investigations have shown that the velocity profile in a turbulent boundary layer, outside the laminar sublayer, can be described by a one-seventh-power relation

$$\frac{u}{u_\infty} = \left(\frac{y}{\delta}\right)^{1/7} \qquad \textbf{[5-88]}$$

where δ is the boundary-layer thickness as before. For purposes of an integral analysis the momentum integral can be evaluated with Equation (5-88) because the laminar sublayer is so thin. However, the wall shear stress cannot be calculated from Equation (5-88) because it yields an infinite value at $y = 0$.

To determine the turbulent-boundary-layer thickness we employ Equation (5-17) for the integral momentum relation and evaluate the wall shear stress from the empirical relations for skin friction presented previously. According to Equation (5-52),

$$\tau_w = \frac{C_f \rho u_\infty^2}{2}$$

and so for $\mathrm{Re}_x < 10^7$ we obtain from Equation (5-77)

$$\tau_w = 0.0296 \left(\frac{\nu}{u_\infty x}\right)^{1/5} \rho u_\infty^2 \qquad \textbf{[5-89]}$$

Now, using the integral momentum equation for zero pressure gradient [Equation (5-17)] along with the velocity profile and wall shear stress, we obtain

$$\frac{d}{dx}\int_0^\delta \left[1 - \left(\frac{y}{\delta}\right)^{1/7}\right]\left(\frac{y}{\delta}\right)^{1/7} dy = 0.0296\left(\frac{\nu}{u_\infty x}\right)^{1/5}$$

Integrating and clearing terms gives

$$\frac{d\delta}{dx} = \frac{72}{7}(0.0296)\left(\frac{\nu}{u_\infty}\right)^{1/5} x^{-1/5} \qquad \textbf{[5-90]}$$

We shall integrate this equation for two physical situations:

1. The boundary layer is fully turbulent from the leading edge of the plate.

2. The boundary layer follows a laminar growth pattern up to $\mathrm{Re}_{crit} = 5 \times 10^5$ and a turbulent growth thereafter.

For the first case, we integrate Equation (5-89) with the condition that $\delta = 0$ at $x = 0$ to obtain

$$\frac{\delta}{x} = 0.381\ \mathrm{Re}_x^{-1/5} \qquad \textbf{[5-91]}$$

For case 2 we have the condition

$$\delta = \delta_{lam} \qquad \text{at } x_{crit} = 5 \times 10^5 \frac{\nu}{u_\infty} \qquad \textbf{[5-92]}$$

Now, δ_{lam} is calculated from the exact relation of Equation (5-21*a*):

$$\delta_{lam} = 5.0 x_{crit} (5 \times 10^5)^{-1/2} \qquad \textbf{[5-93]}$$

Integrating Equation (5-89) gives

$$\delta - \delta_{lam} = \frac{72}{7}(0.0296)\left(\frac{\nu}{u_\infty}\right)^{1/5} \frac{5}{4}\left(x^{4/5} - x_{crit}^{4/5}\right) \qquad \textbf{[5-94]}$$

Combining the various relations above gives

$$\frac{\delta}{x} = 0.381\ \mathrm{Re}_x^{-1/5} - 10{,}256\ \mathrm{Re}_x^{-1} \qquad \textbf{[5-95]}$$

This relation applies only for the region $5 \times 10^5 < \mathrm{Re}_x < 10^7$.

Turbulent-Boundary-Layer Thickness

EXAMPLE 5-10

Calculate the turbulent-boundary-layer thickness at the end of the plate for Example 5-9, assuming that it develops (*a*) from the leading edge of the plate and (*b*) from the transition point at $\mathrm{Re}_{crit} = 5 \times 10^5$.

Solution

Since we have already calculated the Reynolds number as $\mathrm{Re}_L = 1.553 \times 10^6$, it is a simple matter to insert this value in Equations (5-91) and (5-95) along with $x = L = 0.75$ m to give

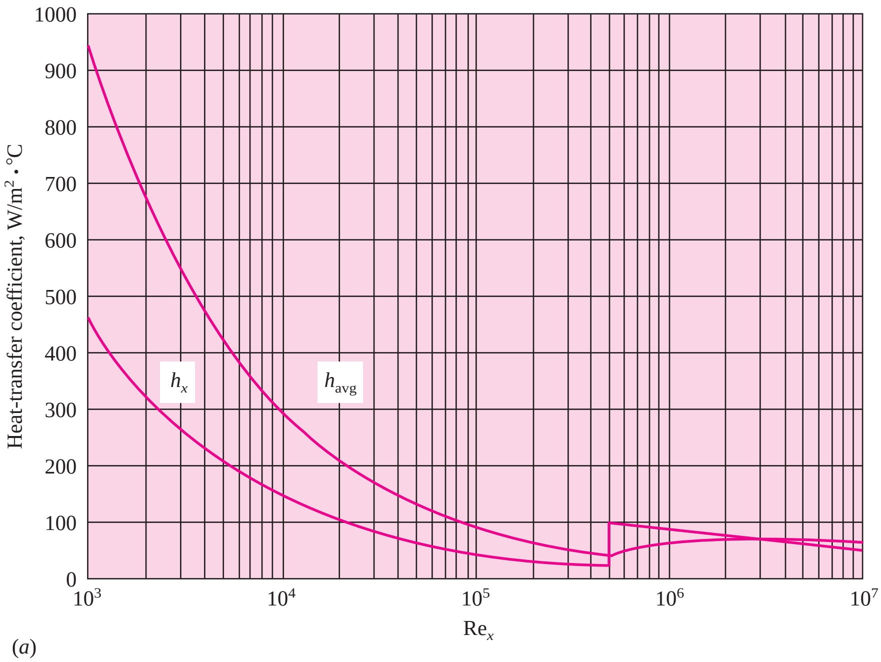
$$(a)\ \delta = (0.75)(0.381)(1.553 \times 10^6)^{-0.2} = 0.0165 \text{ m} = 16.5 \text{ mm } [0.65 \text{ in}]$$

$$(b)\ \delta = (0.75)[(0.381)(1.553 \times 10^6)^{-0.2} - 10{,}256(1.553 \times 10^6)^{-1}]$$
$$= 0.0099 \text{ m} = 9.9 \text{ mm } [0.39 \text{ in}]$$

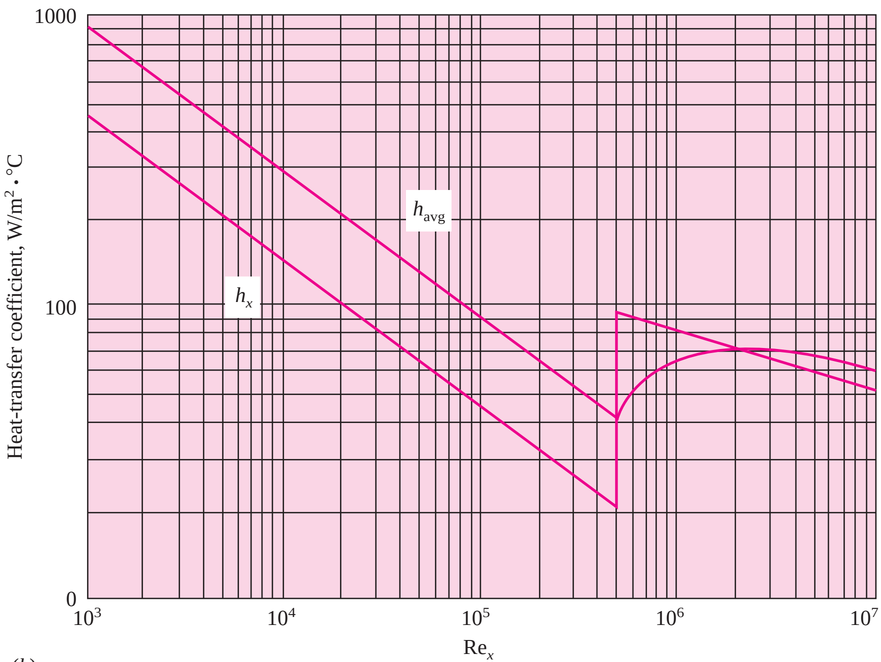
The two values differ by 40 percent.

An overall perspective of the behavior of the local and average heat-transfer coefficients is indicated in Figure 5-13. The fluid is atmospheric air flowing across an isothermal flat plate at $u_\infty = 30$ m/s, and the calculations were made with Equations (5-55), (5-81), and (5-85), which assume a value of $\mathrm{Re}_{crit} = 5 \times 10^5$. The corresponding value of x_{crit} is 0.2615 m and the plate length is 5.23 m at $\mathrm{Re} = 10^7$. The corresponding boundary-layer thickness is plotted in Figure 5-14. As we have noted before, the heat-transfer coefficient varies inversely with the boundary-layer thickness, and an increase in heat transfer is experienced when turbulence begins.

Figure 5-13 | Local and average heat-transfer coefficient for atmospheric airflow over isothermal flat plate at $u_\infty = 30$ m/s (*a*) semilog scale (*b*) log scale.

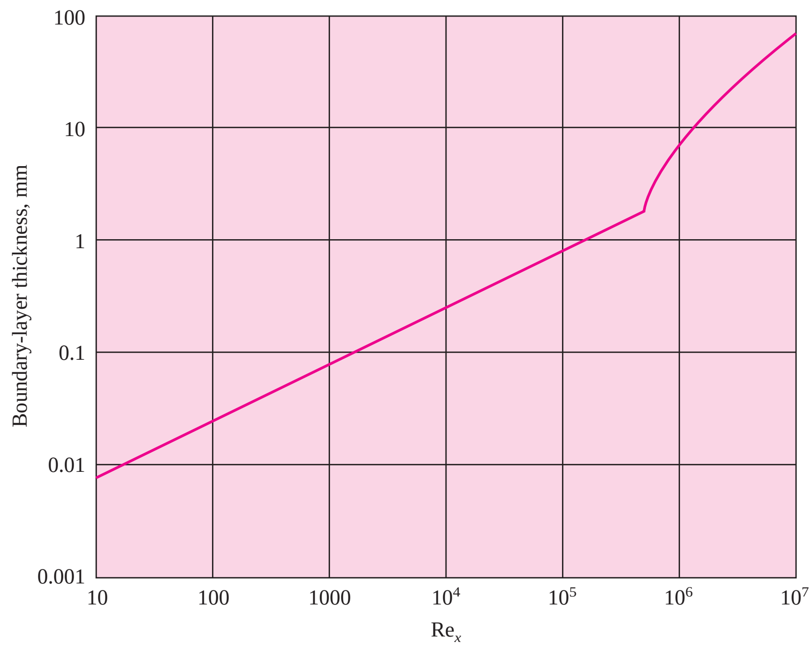


(*a*)



(*b*)

Figure 5-14 | Boundary-layer thickness for atmospheric air at $u_\infty = 30$ m/s.



5-10 | HEAT TRANSFER IN LAMINAR TUBE FLOW

Consider the tube-flow system in Figure 5-15. We wish to calculate the heat transfer under developed flow conditions when the flow remains laminar. The wall temperature is T_w, the radius of the tube is r_o, and the velocity at the center of the tube is u_0. It is assumed that the pressure is uniform at any cross section. The velocity distribution may be derived by considering the fluid element shown in Figure 5-16. The pressure forces are balanced by the viscous-shear forces so that

$$\pi r^2\, dp = \tau 2\pi r\, dx = 2\pi r \mu\, dx\, \frac{du}{dr}$$

or

$$du = \frac{1}{2\mu} r \frac{dp}{dx}\, dr$$

Figure 5-15 | Control volume for energy analysis in tube flow.

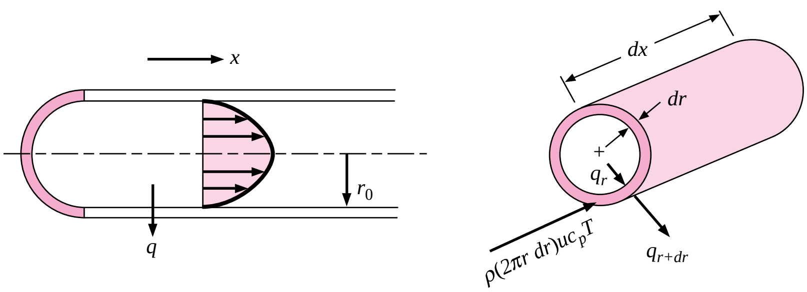
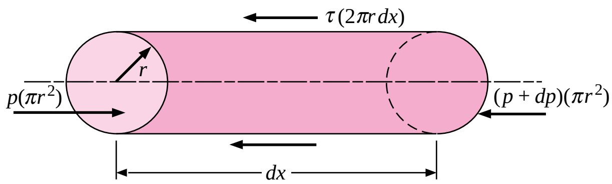


Figure 5-16 | Force balance on fluid element in tube flow.



and

$$u = \frac{1}{4\mu}\frac{dp}{dx}r^2 + \text{const} \qquad \textbf{[5-96]}$$

With the boundary condition

$$u = 0 \qquad \text{at } r = r_o$$

$$u = \frac{1}{4\mu}\frac{dp}{dx}(r^2 - r_o^2)$$

the velocity at the center of the tube is given by

$$u_0 = -\frac{r_o^2}{4\mu}\frac{dp}{dx} \qquad \textbf{[5-97]}$$

so that the velocity distribution may be written

$$\frac{u}{u_0} = 1 - \frac{r^2}{r_o^2} \qquad \textbf{[5-98]}$$

which is the familiar parabolic distribution for laminar tube flow. Now consider the heat-transfer process for such a flow system. To simplify the analysis, we assume that there is a constant heat flux at the tube wall; that is,

$$\frac{dq_w}{dx} = 0$$

The heat flow conducted into the annular element is

$$dq_r = -k2\pi r\,dx\,\frac{\partial T}{\partial r}$$

and the heat conducted out is

$$dq_{r+dr} = -k2\pi(r+dr)\,dx\left(\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial r^2}\,dr\right)$$

The net heat convected out of the element is

$$2\pi r\,dr\,\rho c_p u\frac{\partial T}{\partial x}\,dx$$

The energy balance is

Net energy convected out = net heat conducted in

or, neglecting second-order differentials,

$$r\rho c_p u\frac{\partial T}{\partial x}\,dx\,dr = k\left(\frac{\partial T}{\partial r} + r\frac{\partial^2 T}{\partial r^2}\right)dx\,dr$$

which may be rewritten

$$\frac{1}{ur}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = \frac{1}{\alpha}\frac{\partial T}{\partial x} \tag{5-99}$$

We assume that the heat flux at the wall is constant, so that the average fluid temperature must increase linearly with x, or

$$\frac{\partial T}{\partial x} = \text{const}$$

This means that the temperature profiles will be similar at various x distances along the tube. The boundary conditions on Equation (5-98) are

$$\frac{\partial T}{\partial r} = 0 \qquad \text{at } r = 0$$

$$k\frac{\partial T}{\partial r}\bigg]_{r=r_o} = q_w = \text{const}$$

To obtain the solution to Equation (5-99), the velocity distribution given by Equation (5-98) must be inserted. It is assumed that the temperature and velocity fields are independent; that is, a temperature gradient does not affect the calculation of the velocity profile. This is equivalent to specifying that the properties remain constant in the flow. With the substitution of the velocity profile, Equation (5-99) becomes

$$\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = \frac{1}{\alpha}\frac{\partial T}{\partial x}u_0\left(1 - \frac{r^2}{r_o^2}\right) r$$

Integration yields

$$r\frac{\partial T}{\partial r} = \frac{1}{\alpha}\frac{\partial T}{\partial x}u_0\left(\frac{r^2}{2} - \frac{r^4}{4r_o^2}\right) + C_1$$

and a second integration gives

$$T = \frac{1}{\alpha}\frac{\partial T}{\partial x}u_0\left(\frac{r^2}{4} - \frac{r^4}{16r_o^2}\right) + C_1 \ln r + C_2$$

Applying the first boundary condition, we find that

$$C_1 = 0$$

The second boundary condition has been satisfied by noting that the axial temperature gradient $\partial T/\partial x$ is constant. The temperature distribution may finally be written in terms of the temperature at the center of the tube:

$$T = T_c \qquad \text{at } r = 0 \qquad \text{so that} \qquad C_2 = T_c$$

$$T - T_c = \frac{1}{\alpha}\frac{\partial T}{\partial x}\frac{u_0 r_o^2}{4}\left[\left(\frac{r}{r_o}\right)^2 - \frac{1}{4}\left(\frac{r}{r_o}\right)^4\right] \tag{5-100}$$

The Bulk Temperature

In tube flow the convection heat-transfer coefficient is usually defined by

$$\text{Local heat flux} = q'' = h(T_w - T_b) \tag{5-101}$$

where T_w is the wall temperature and T_b is the so-called *bulk temperature*, or energy-average fluid temperature across the tube, which may be calculated from

$$T_b = \overline{T} = \frac{\int_0^{r_o} \rho 2\pi r\, dr\, u c_p T}{\int_0^{r_o} \rho 2\pi r\, dr\, u c_p} \tag{5-102}$$

The reason for using the bulk temperature in the definition of heat-transfer coefficients for tube flow may be explained as follows. In a tube flow there is no easily discernible free-stream condition as is present in the flow over a flat plate. Even the centerline temperature T_c is not easily expressed in terms of the inlet flow variables and the heat transfer. For most tube- or channel-flow heat-transfer problems, the topic of central interest is the total energy transferred to the fluid in either an elemental length of the tube or over the entire length of the channel. At any x position, the temperature that is indicative of the total energy of the flow is an integrated mass-energy average temperature over the entire flow area. The numerator of Equation (5-102) represents the total energy flow through the tube, and the denominator represents the product of mass flow and specific heat integrated over the flow area. The bulk temperature is thus representative of the total energy of the flow at the particular location. For this reason, the bulk temperature is sometimes referred to as the "mixing cup" temperature, since it is the temperature the fluid would assume if placed in a mixing chamber and allowed to come to equilibrium. For the temperature distribution given in Equation (5-100), the bulk temperature is a linear function of x because the heat flux at the tube wall is constant. Calculating the bulk temperature from Equation (5-102), we have

$$T_b = T_c + \frac{7}{96}\frac{u_0 r_o^2}{\alpha}\frac{\partial T}{\partial x} \tag{5-103}$$

and for the wall temperature

$$T_w = T_c + \frac{3}{16}\frac{u_0 r_o^2}{\alpha}\frac{\partial T}{\partial x} \tag{5-104}$$

The heat-transfer coefficient is calculated from

$$q = hA(T_w - T_b) = kA\left(\frac{\partial T}{\partial r}\right)_{r=r_o} \tag{5-105}$$

$$h = \frac{k(\partial T/\partial r)_{r=r_o}}{T_w - T_b}$$

The temperature gradient is given by

$$\left.\frac{\partial T}{\partial r}\right]_{r=r_o} = \frac{u_0}{\alpha}\frac{\partial T}{\partial x}\left(\frac{r}{2} - \frac{r^3}{4r_o^2}\right)_{r=r_o} = \frac{u_0 r_o}{4\alpha}\frac{\partial T}{\partial x} \tag{5-106}$$

Substituting Equations (5-103), (5-104), and (5-106) in Equation (5-105) gives

$$h = \frac{24}{11}\frac{k}{r_o} = \frac{48}{11}\frac{k}{d_o}$$

Expressed in terms of the Nusselt number, the result is

$$\mathrm{Nu}_d = \frac{h d_o}{k} = 4.364 \tag{5-107}$$

which is in agreement with an exact calculation by Sellars, Tribus, and Klein [3], that considers the temperature profile as it develops. Some empirical relations for calculating heat transfer in laminar tube flow will be presented in Chapter 6.

We may remark at this time that when the statement is made that a fluid enters a tube at a certain temperature, it is the bulk temperature to which we refer. The bulk temperature is used for overall energy balances on systems.

5-11 | TURBULENT FLOW IN A TUBE

The developed velocity profile for turbulent flow in a tube will appear as shown in Figure 5-17. A laminar sublayer, or "film," occupies the space near the surface, while the central core of the flow is turbulent. To determine the heat transfer analytically for this situation, we require, as usual, a knowledge of the temperature distribution in the flow. To obtain this temperature distribution, the analysis must take into consideration the effect of the turbulent eddies in the transfer of heat and momentum. We shall use an approximate analysis that relates the conduction and transport of heat to the transport of momentum in the flow (i.e., viscous effects).

The heat flow across a fluid element in laminar flow may be expressed by

$$\frac{q}{A} = -k\frac{dT}{dy}$$

Dividing both sides of the equation by ρc_p,

$$\frac{q}{\rho c_p A} = -\alpha\frac{dT}{dy}$$

It will be recalled that α is the molecular diffusivity of heat. In turbulent flow one might assume that the heat transport could be represented by

$$\frac{q}{\rho c_p A} = -(\alpha + \epsilon_H)\frac{dT}{dy} \tag{5-108}$$

where ϵ_H is an eddy diffusivity of heat.

Equation (5-108) expresses the total heat conduction as a sum of the molecular conduction and the macroscopic eddy conduction. In a similar fashion, the shear stress in turbulent flow could be written

$$\frac{\tau}{\rho} = \left(\frac{\mu}{\rho} + \epsilon_M\right)\frac{du}{dy} = (\nu + \epsilon_M)\frac{du}{dy} \tag{5-109}$$

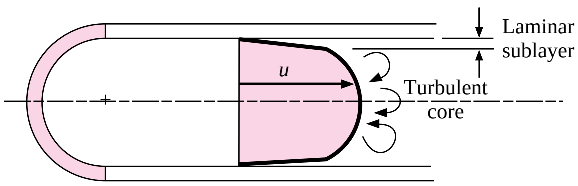
where ϵ_M is the eddy diffusivity for momentum. We now assume that the heat and momentum are transported at the same rate; that is, $\epsilon_M = \epsilon_H$ and $\nu = \alpha$, or $\Pr = 1$.

Dividing Equation (5-108) by Equation (5-109) gives

$$\frac{q}{c_p A\tau}\,du = -dT$$

An additional assumption is that the ratio of the heat transfer per unit area to the shear stress is constant across the flow field. This is consistent with the assumption that heat and

Figure 5-17 | Velocity profile in turbulent tube flow.



momentum are transported at the same rate. Thus

$$\frac{q}{A\tau} = \text{const} = \frac{q_w}{A_w \tau_w} \qquad \textbf{[5-110]}$$

Then, integrating Equation (5-109) between wall conditions and mean bulk conditions gives

$$\frac{q_w}{A_w \tau_w c_p} \int_{u=0}^{u=u_m} du = \int_{T_w}^{T_b} -dT$$

$$\frac{q_w u_m}{A_w \tau_w c_p} = T_w - T_b \qquad \textbf{[5-111]}$$

But the heat transfer at the wall may be expressed by

$$q_w = h A_w (T_w - T_b)$$

and the shear stress may be calculated from

$$\tau_w = \frac{\Delta p(\pi d_o^2)}{4\pi d_o L} = \frac{\Delta p}{4} \frac{d_o}{L}$$

The pressure drop may be expressed in terms of a friction factor f by

$$\Delta p = f \frac{L}{d_o} \rho \frac{u_m^2}{2} \qquad \textbf{[5-112]}$$

so that

$$\tau_w = \frac{f}{8} \rho u_m^2 \qquad \textbf{[5-113]}$$

Substituting the expressions for τ_w and q_w in Equation (5-111) gives

$$\text{St} = \frac{h}{\rho c_p u_m} = \frac{\text{Nu}_d}{\text{Re}_d \, \text{Pr}} = \frac{f}{8} \qquad \textbf{[5-114]}$$

Equation (5-114) is called the Reynolds analogy for tube flow. It relates the heat-transfer rate to the frictional loss in tube flow and is in fair agreement with experiments when used with gases whose Prandtl numbers are close to unity. (Recall that $\text{Pr} = 1$ was one of the assumptions in the analysis.)

An empirical formula for the turbulent-friction factor up to Reynolds numbers of about 2×10^5 for the flow in smooth tubes is

$$f = \frac{0.316}{\text{Re}_d^{1/4}} \qquad \textbf{[5-115]}$$

Inserting this expression in Equation (5-~~113~~) gives

$$\frac{\text{Nu}_d}{\text{Re}_d \, \text{Pr}} = 0.0395 \, \text{Re}_d^{-1/4}$$

or

$$\text{Nu}_d = 0.0395 \, \text{Re}_d^{3/4} \qquad \textbf{[5-116]}$$

since we assumed the Prandtl number to be unity. This derivation of the relation for turbulent heat transfer in smooth tubes is highly restrictive because of the $\text{Pr} \approx 1.0$ assumption. The heat-transfer–fluid-friction analogy of Section 5-7 indicated a Prandtl-number dependence of $\mathrm{Pr}^{2/3}$ for the flat-plate problem and, as it turns out, this dependence works fairly well for turbulent tube flow. Equations (5-114) and (5-116) may be modified by this factor to yield

$$\mathrm{St}\,\mathrm{Pr}^{2/3} = \frac{f}{8} \tag{5-114a}$$

$$\mathrm{Nu}_d = 0.0395\,\mathrm{Re}_d^{3/4}\,\mathrm{Pr}^{1/3} \tag{5-116a}$$

As we shall see in Chapter 6, Equation (5-116*a*) predicts heat-transfer coefficients that are somewhat higher than those observed in experiments. The purpose of the discussion at this point has been to show that one may arrive at a relation for turbulent heat transfer in a fairly simple analytical fashion. As we have indicated earlier, a rigorous development of the Reynolds analogy between heat transfer and fluid friction involves considerations beyond the scope of our discussion, and the simple path of reasoning chosen here is offered for the purpose of indicating the general nature of the physical processes.

For calculation purposes, a more correct relation to use for turbulent flow in a smooth tube is Equation (6-4a), which we list here for comparison:

$$\mathrm{Nu}_d = 0.023\,\mathrm{Re}_d^{0.8}\,\mathrm{Pr}^{0.4} \tag{6-4a}$$

All properties in Equation (6-4*a*) are evaluated at the bulk temperature.

5-12 | HEAT TRANSFER IN HIGH-SPEED FLOW

Our previous analysis of boundary-layer heat transfer (Section 5-6) neglected the effects of viscous dissipation within the boundary layer. When the free-stream velocity is very high, as in high-speed aircraft, these dissipation effects must be considered. We begin our analysis by considering the adiabatic case, i.e., a perfectly insulated wall. In this case the wall temperature may be considerably higher than the free-stream temperature even though no heat transfer takes place. This high temperature results from two situations: (1) the increase in temperature of the fluid as it is brought to rest at the plate surface while the kinetic energy of the flow is converted to internal thermal energy, and (2) the heating effect due to viscous dissipation. Consider the first situation. The kinetic energy of the gas is converted to thermal energy as the gas is brought to rest, and this process is described by the steady-flow energy equation for an adiabatic process:

$$i_0 = i_\infty + \frac{1}{2g_c}u_\infty^2 \tag{5-117}$$

where i_0 is the stagnation enthalpy of the gas. This equation may be written in terms of temperature as

$$c_p(T_0 - T_\infty) = \frac{1}{2g_c}u_\infty^2$$

where T_0 is the stagnation temperature and T_∞ is the static free-stream temperature. Expressed in terms of the free-stream Mach number, it is

$$\frac{T_0}{T_\infty} = 1 + \frac{\gamma - 1}{2}M_\infty^2 \tag{5-118}$$

where M_∞ is the Mach number, defined as $M_\infty = u_\infty/a$, and a is the acoustic velocity, which for an ideal gas may be calculated with

$$a = \sqrt{\gamma g_c R T} \tag{5-119}$$

where R is the gas constant for the particular gas.

In the actual case of a boundary-layer flow problem, the fluid is not brought to rest reversibly because the viscous action is basically an irreversible process in a thermodynamic sense. In addition, not all the free-stream kinetic energy is converted to thermal energy—part is lost as heat, and part is dissipated in the form of viscous work. To take into account the irreversibilities in the boundary-layer flow system, a *recovery factor* is defined by

$$r = \frac{T_{aw} - T_\infty}{T_0 - T_\infty} \tag{5-120}$$

where T_{aw} is the actual adiabatic wall temperature and T_∞ is the static temperature of the free stream. The recovery factor may be determined experimentally, or, for some flow systems, analytical calculations may be made.

The boundary-layer energy equation

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} + \frac{\mu}{\rho c_p}\left(\frac{\partial u}{\partial y}\right)^2$$

has been solved for the high-speed-flow situation, taking into account the viscous-heating term. Although the complete solution is somewhat tedious, the final results are remarkably simple. For our purposes we present only the results and indicate how they may be applied. The reader is referred to Appendix B for an exact solution to Equation (5-22). An excellent synopsis of the high-speed heat-transfer problem is given in a report by Eckert [4]. Some typical boundary-layer temperature profiles for an adiabatic wall in high-speed flow are given in Figure B-3.

The essential result of the high-speed heat-transfer analysis is that heat-transfer rates may generally be calculated with the same relations used for low-speed incompressible flow when the average heat-transfer coefficient is redefined with the relation

$$q = \bar{h}A(T_w - T_{aw}) \tag{5-121}$$

Notice that the difference between the adiabatic wall temperature and the actual wall temperature is used in the definition so that the expression will yield a value of zero heat flow when the wall is at the adiabatic wall temperature. For gases with Prandtl numbers near unity, the following relations for the recovery factor have been derived:

Laminar flow:
$$r = \mathrm{Pr}^{1/2} \tag{5-122}$$

Turbulent flow:
$$r = \mathrm{Pr}^{1/3} \tag{5-123}$$

These recovery factors may be used in conjunction with Equation (5-120) to obtain the adiabatic wall temperature.

In high-velocity boundary layers substantial temperature gradients may occur, and there will be correspondingly large property variations across the boundary layer. The constant-property heat-transfer equations may still be used if the properties are introduced at a reference temperature T^* as recommended by Eckert:

$$T^* = T_\infty + 0.50(T_w - T_\infty) + 0.22(T_{aw} - T_\infty) \tag{5-124}$$

The analogy between heat transfer and fluid friction [Equation (5-56)] may also be used when the friction coefficient is known. Summarizing the relations used for high-speed heat-transfer calculations:

Laminar boundary layer ($\mathrm{Re}_x < 5 \times 10^5$):

$$\mathrm{St}_x^* \, \mathrm{Pr}^{*2/3} = 0.332 \, \mathrm{Re}_x^{*-1/2} \tag{5-125}$$

Turbulent boundary layer $(5 \times 10^5 < \mathrm{Re}_x < 10^7)$:

$$\mathrm{St}_x^* \, \mathrm{Pr}^{*2/3} = 0.0296 \, \mathrm{Re}_x^{*-1/5} \qquad \textbf{[5-126]}$$

Turbulent boundary layer $(10^7 < \mathrm{Re}_x < 10^9)$:

$$\mathrm{St}_x^* \, \mathrm{Pr}^{*2/3} = 0.185(\log \mathrm{Re}_x^*)^{-2.584} \qquad \textbf{[5-127]}$$

The superscript * in the above equations indicates that the properties are evaluated at the reference temperature given by Equation (5-124).

To obtain an average heat-transfer coefficient, the above expressions must be integrated over the length of the plate. If the Reynolds number falls in a range such that Equation (5-127) must be used, the integration cannot be expressed in closed form, and a numerical integration must be performed. Care must be taken in performing the integration for the high-speed heat-transfer problem since the reference temperature is different for the laminar and turbulent portions of the boundary layer. This results from the different value of the recovery factor used for laminar and turbulent flow as given by Equations (5-122) and (5-123).

When very high flow velocities are encountered, the adiabatic wall temperature may become so high that dissociation of the gas will take place and there will be a very wide variation of the properties in the boundary layer. Eckert [4] recommends that these problems be treated on the basis of a heat-transfer coefficient defined in terms of *enthalpy* difference:

$$q = h_i A(i_w - i_{aw}) \qquad \textbf{[5-128]}$$

The enthalpy recovery factor is then defined as

$$r_i = \frac{i_{aw} - i_\infty}{i_0 - i_\infty} \qquad \textbf{[5-129]}$$

where i_{aw} is the enthalpy at the adiabatic wall conditions. The same relations as before are used to calculate the recovery factor and heat-transfer except that all properties are evaluated at a reference enthalpy i^* given by

$$i^* = i_\infty + 0.5(i_w - i_\infty) + 0.22(i_{aw} - i_\infty) \qquad \textbf{[5-130]}$$

The Stanton number is redefined as

$$\mathrm{St}_i = \frac{h_i}{\rho u_\infty} \qquad \textbf{[5-131]}$$

This Stanton number is then used in Equation (5-125), (5-126), or (5-127) to calculate the heat-transfer coefficient. When calculating the enthalpies for use in the above relations, the *total* enthalpy must be used; that is chemical energy of dissociation as well as internal thermal energy must be included. The reference-enthalpy method has proved successful for calculating high-speed heat-transfer with an accuracy of better than 10 percent.

High-Speed Heat Transfer for a Flat Plate

EXAMPLE 5-11

A flat plate 70 cm long and 1.0 m wide is placed in a wind tunnel where the flow conditions are $M = 3$, $p = \frac{1}{20}$ atm, and $T = -40°\mathrm{C}$. How much cooling must be used to maintain the plate temperature at 35°C?

■ Solution

We must consider the laminar and turbulent portions of the boundary layer separately because the recovery factors, and hence the adiabatic wall temperatures, used to establish the heat flow will be different for each flow regime. It turns out that the difference is rather small in this problem, but we shall follow a procedure that would be used if the difference were appreciable, so that the general method of solution may be indicated. The free-stream acoustic velocity is calculated from

$$a = \sqrt{\gamma g_c R T_\infty} = [(1.4)(1.0)(287)(233)]^{1/2} = 306 \text{ m/s} \quad [1003 \text{ ft/s}]$$

so that the free-stream velocity is

$$u_\infty = (3)(306) = 918 \text{ m/s} \quad [3012 \text{ ft/s}]$$

The maximum Reynolds number is estimated by making a computation based on properties evaluated at free-stream conditions:

$$\rho_\infty = \frac{(1.0132 \times 10^5)(\frac{1}{20})}{(287)(233)} = 0.0758 \text{ kg/m}^3 \quad [4.73 \times 10^{-3} \text{ lb}_m/\text{ft}^3]$$

$$\mu_\infty = 1.434 \times 10^{-5} \text{ kg/m} \cdot \text{s} \quad [0.0347 \text{ lb}_m/\text{h} \cdot \text{ft}]$$

$$\text{Re}_{L,\infty} = \frac{(0.0758)(918)(0.70)}{1.434 \times 10^{-5}} = 3.395 \times 10^6$$

Thus we conclude that both laminar and turbulent-boundary-layer heat transfer must be considered. We first determine the reference temperatures for the two regimes and then evaluate properties at these temperatures.

Laminar portion

$$T_0 = T_\infty \left(1 + \frac{\gamma - 1}{2} M_\infty^2\right) = (233)[1 + (0.2)(3)^2] = 652 \text{ K}$$

Assuming a Prandtl number of about 0.7, we have

$$r = \text{Pr}^{1/2} = (0.7)^{1/2} = 0.837$$

$$r = \frac{T_{aw} - T_\infty}{T_0 - T_\infty} = \frac{T_{aw} - 233}{652 - 233}$$

and $T_{aw} = 584$ K $= 311°$C [592°F]. Then the reference temperature from Equation (5-123) is

$$T^* = 233 + (0.5)(308 - 233) + (0.22)(584 - 233) = 347.8 \text{ K}$$

Checking the Prandtl number at this temperature, we have

$$\text{Pr}^* = 0.697$$

so that the calculation is valid. If there were an appreciable difference between the value of Pr* and the value used to determine the recovery factor, the calculation would have to be repeated until agreement was reached.

The other properties to be used in the laminar heat-transfer analysis are

$$\rho^* = \frac{(1.0132 \times 10^5)(1/20)}{(287)(347.8)} = 0.0508 \text{ kg/m}^3$$

$$\mu^* = 2.07 \times 10^{-5} \text{ kg/m} \cdot \text{s}$$

$$k^* = 0.03 \text{ W/m} \cdot °\text{C} \quad [0.0173 \text{ Btu/h} \cdot \text{ft} \cdot °\text{F}]$$

$$c_p^* = 1.009 \text{ kJ/kg} \cdot °\text{C}$$

Turbulent portion

Assuming Pr = 0.7 gives

$$r = \mathrm{Pr}^{1/3} = 0.888 = \frac{T_{aw} - T_\infty}{T_0 - T_\infty} = \frac{T_{aw} - 233}{652 - 233}$$

$$T_{aw} = 605\ \mathrm{K} = 332^\circ\mathrm{C}$$

$$T^* = 233 + (0.5)(308 - 233) + (0.22)(605 - 233) = 352.3\ \mathrm{K}$$

$$\mathrm{Pr}^* = 0.695$$

The agreement between Pr* and the assumed value is sufficiently close. The other properties to be used in the turbulent heat-transfer analysis are

$$\rho^* = \frac{(1.0132 \times 10^5)(1/20)}{(287)(352.3)} = 0.0501\ \mathrm{kg/m^3}$$

$$\mu^* = 2.09 \times 10^{-5}\ \mathrm{kg/m \cdot s}$$

$$k^* = 0.0302\ \mathrm{W/m \cdot {}^\circ C} \qquad c_p{}^* = 1.009\ \mathrm{kJ/kg \cdot {}^\circ C}$$

Laminar heat transfer

We assume

$$\mathrm{Re}^*_{\mathrm{crit}} = 5 \times 10^5 = \frac{\rho^* u_\infty x_c}{\mu^*}$$

$$x_c = \frac{(5 \times 10^5)(2.07 \times 10^{-5})}{(0.0508)(918)} = 0.222\ \mathrm{m}$$

$$\overline{\mathrm{Nu}^*} = \frac{\bar{h} x_c}{k^*} = 0.664 \left(\mathrm{Re}^*_{\mathrm{crit}}\right)^{1/2}\ \mathrm{Pr}^{*1/3}$$

$$= (0.664)(5 \times 10^5)^{1/2}(0.697)^{1/3} = 416.3$$

$$\bar{h} = \frac{(416.3)(0.03)}{0.222} = 56.25\ \mathrm{W/m^2 \cdot {}^\circ C} \quad [9.91\ \mathrm{Btu/h \cdot ft^2 \cdot {}^\circ F}]$$

This is the average heat-transfer coefficient for the laminar portion of the boundary layer, and the heat transfer is calculated from

$$q = \bar{h} A (T_w - T_{aw})$$

$$= (56.26)(0.222)(308 - 584)$$

$$= -3445\ \mathrm{W} \quad [-11{,}750\ \mathrm{Btu/h}]$$

so that 3445 W of cooling is required in the laminar region of the plate per meter of depth in the z direction.

Turbulent heat transfer

To determine the turbulent heat transfer we must obtain an expression for the local heat-transfer coefficient from

$$\mathrm{St}^*_x\ \mathrm{Pr}^{*2/3} = 0.0296\ \mathrm{Re}_x^{*-1/5}$$

and then integrate from $x = 0.222$ m to $x = 0.7$ m to determine the total heat transfer:

$$h_x = \mathrm{Pr}^{*-2/3} \rho^* u_\infty c_p (0.0296) \left(\frac{\rho^* u_\infty x}{\mu^*}\right)^{-1/5}$$

Inserting the numerical values for the properties gives

$$h_x = 94.34 x^{-1/5}$$

The average heat-transfer coefficient in the turbulent region is determined from

$$\bar{h} = \frac{\int_{0.222}^{0.7} h_x \, dx}{\int_{0.222}^{0.7} dx} = 111.46 \text{ W/m}^2 \cdot {}^\circ\text{C} \quad [19.6 \text{ Btu/h} \cdot \text{ft}^2 \cdot {}^\circ\text{F}]$$

Using this value we may calculate the heat transfer in the turbulent region of the flat plate:

$$\begin{aligned} q &= \bar{h} A (T_w - T_{aw}) \\ &= (111.46)(0.7 - 0.222)(308 - 605) \\ &= -15{,}823 \text{ W} \quad [-54{,}006 \text{ Btu/h}] \end{aligned}$$

The total amount of cooling required is the sum of the heat transfers for the laminar and turbulent portions:

$$\text{Total cooling} = 3445 + 15{,}823 = 19{,}268 \text{ W} \quad [65{,}761 \text{ Btu/h}]$$

These calculations assume unit depth of 1 m in the z direction.

5-13 | SUMMARY

Most of this chapter has been concerned with flow over flat plates and the associated heat transfer. For convenience of the reader we have summarized the heat-transfer, boundary-layer thickness, and friction-coefficient equations in Table 5-2 along with the restrictions that apply. Our presentation of convection heat transfer is incomplete at this time and will be developed further in Chapters 6 and 7. Even so, we begin to see the structure of a procedure for solution of convection problems:

1. Establish the geometry of the situation; for now we are mainly restricted to flow over flat plates.
2. Determine the fluid involved and evaluate the fluid properties. This will usually be at the film temperature.
3. Establish the boundary conditions (i.e., constant temperature or constant heat flux).
4. Establish the flow regime as determined by the Reynolds number.
5. Select the appropriate equation, taking into account the flow regime and any fluid property restrictions which may apply.
6. Calculate the value(s) of the convection heat-transfer coefficient and/or heat transfer.

At the conclusion of Chapter 7 we shall give a general procedure for all convection problems and the information contained in Table 5-2 will comprise one ingredient in the overall recipe. The interested reader may wish to consult Section 7-14 and Figure 7-15 for a preview of this information and some perspective of the way the material in the present chapter fits in.

REVIEW QUESTIONS

1. What is meant by a hydrodynamic boundary level?
2. Define the Reynolds number. Why is it important?
3. What is the physical mechanism of viscous action?
4. Distinguish between laminar and turbulent flow in a physical sense.

Table 5-2 | Summary of equations for flow over flat plates. Properties evaluated at $T_f = (T_w + T_\infty)/2$ unless otherwise noted.

Flow regime	Restrictions	Equation	Equation number
		Heat transfer	
Laminar, local	$T_w = \text{const}$, $\text{Re}_x < 5 \times 10^5$, $0.6 < \text{Pr} < 50$	$\text{Nu}_x = 0.332\ \text{Pr}^{1/3}\text{Re}_x^{1/2}$	(5-44)
Laminar, local	$T_w = \text{const}$, $\text{Re}_x < 5 \times 10^5$, $\text{Re}_x\ \text{Pr} > 100$	$\text{Nu}_x = \dfrac{0.3387\ \text{Re}_x^{1/2}\ \text{Pr}^{1/3}}{\left[1 + \left(\dfrac{0.0468}{\text{Pr}}\right)^{2/3}\right]^{1/4}}$	(5-51)
Laminar, local	$q_w = \text{const}$, $\text{Re}_x < 5 \times 10^5$, $0.6 < \text{Pr} < 50$	$\text{Nu}_x = 0.453\ \text{Re}_x^{1/2}\ \text{Pr}^{1/3}$	(5-48)
Laminar, local	$q_w = \text{const}$, $\text{Re}_x < 5 \times 10^5$	$\text{Nu}_x = \dfrac{0.4637\ \text{Re}_x^{1/2}\ \text{Pr}^{1/3}}{\left[1 + \left(\dfrac{0.0207}{\text{Pr}}\right)^{2/3}\right]^{1/4}}$	(5-51)
Laminar, average	$\text{Re}_L < 5 \times 10^5$, $T_w = \text{const}$	$\overline{\text{Nu}}_L = 2\ \text{Nu}_{x=L} = 0.664\ \text{Re}_L^{1/2}\ \text{Pr}^{1/3}$	(5-46)
Laminar, local	$T_w = \text{const}$, $\text{Re}_x < 5 \times 10^5$, $\text{Pr} \ll 1$ (liquid metals)	$\text{Nu}_x = 0.564(\text{Re}_x\ \text{Pr})^{1/2}$	
Laminar, local	$T_w = \text{const}$, starting at $x = x_0$, $\text{Re}_x < 5 \times 10^5$, $0.6 < \text{Pr} < 50$	$\text{Nu}_x = 0.332\ \text{Pr}^{1/3}\ \text{Re}_x^{1/2}\left[1 - \left(\frac{x_0}{x}\right)^{3/4}\right]^{-1/3}$	(5-43)
Turbulent, local	$T_w = \text{const}$, $5 \times 10^5 < \text{Re}_x < 10^7$	$\text{St}_x\ \text{Pr}^{2/3} = 0.0296\ \text{Re}_x^{-0.2}$	(5-81)
Turbulent, local	$T_w = \text{const}$, $10^7 < \text{Re}_x < 10^9$	$\text{St}_x\ \text{Pr}^{2/3} = 0.185(\log \text{Re}_x)^{-2.584}$	(5-82)
Turbulent, local	$q_w = \text{const}$, $5 \times 10^5 < \text{Re}_x < 10^7$	$\text{Nu}_x = 1.04\ \text{Nu}_{x\,T_w=\text{const}}$	(5-87)
Laminar-turbulent, average	$T_w = \text{const}$, $\text{Re}_x < 10^7$, $\text{Re}_{\text{crit}} = 5 \times 10^5$	$\overline{\text{St}}\ \text{Pr}^{2/3} = 0.037\ \text{Re}_L^{-0.2} - 871\ \text{Re}_L^{-1}$ $\overline{\text{Nu}}_L = \text{Pr}^{1/3}(0.037\ \text{Re}_L^{0.8} - 871)$	(5-84) (5-85)
Laminar-turbulent, average	$T_w = \text{const}$, $\text{Re}_x < 10^7$, liquids, μ at T_∞, μ_w at T_w	$\overline{\text{Nu}}_L = 0.036\ \text{Pr}^{0.43}(\text{Re}_L^{0.8} - 9200)\left(\dfrac{\mu_\infty}{\mu_w}\right)^{1/4}$	(5-86)
High-speed flow	$T_w = \text{const}$, $q = hA(T_w - T_{aw})$ $r = (T_{aw} - T_\infty)/(T_o - T_\infty)$ $=$ recovery factor $= \text{Pr}^{1/2}$ (laminar) $= \text{Pr}^{1/3}$ (turbulent)	Same as for low-speed flow with properties evaluated at $T^* = T_\infty + 0.5(T_w - T_\infty) + 0.22(T_{aw} - T_\infty)$	(5-124)
		Boundary-layer thickness	
Laminar	$\text{Re}_x < 5 \times 10^5$	$\frac{\delta}{x} = 5.0\ \text{Re}_x^{-1/2}$	(5-21*a*)
Turbulent	$\text{Re}_x < 10^7$, $\delta = 0$ at $x = 0$	$\frac{\delta}{x} = 0.381\ \text{Re}_x^{-1/5}$	(5-91)
Turbulent	$5 \times 10^5 < \text{Re}_x < 10^7$, $\text{Re}_{\text{crit}} = 5 \times 10^5$, $\delta = \delta_{\text{lam}}$ at Re_{crit}	$\frac{\delta}{x} = 0.381\ \text{Re}_x^{-1/5} - 10{,}256\ \text{Re}_x^{-1}$	(5-95)
		Friction coefficients	
Laminar, local	$\text{Re}_x < 5 \times 10^5$	$C_{fx} = 0.332\ \text{Re}_x^{-1/2}$	(5-54)
Turbulent, local	$5 \times 10^5 < \text{Re}_x < 10^7$	$C_{fx} = 0.0592\ \text{Re}_x^{-1/5}$	(5-77)
Turbulent, local	$10^7 < \text{Re}_x < 10^9$	$C_{fx} = 0.37(\log \text{Re}_x)^{-2.584}$	(5-78)
Turbulent, average	$\text{Re}_{\text{crit}} < \text{Re}_x < 10^9$	$\overline{C}_f = \frac{0.455}{(\log \text{Re}_L)^{2.584}} - \frac{A}{\text{Re}_L}$ A from Table 5-1	(5-79)